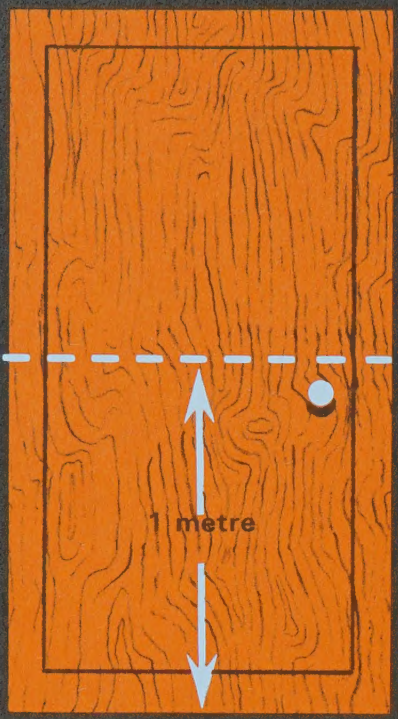
Book 3
Second Edition

mathematics for a modern world

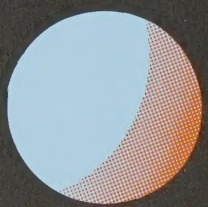
Burns
Pinkney
Del Grande

SI: system of



1 metre (1 m) is about half the height of a door

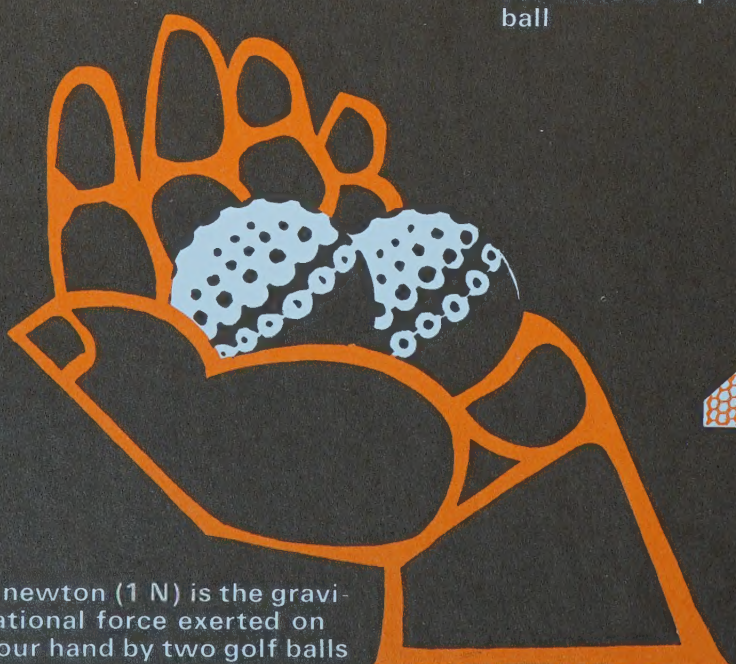
1 millimetre (1 mm) is about the thickness of a dime



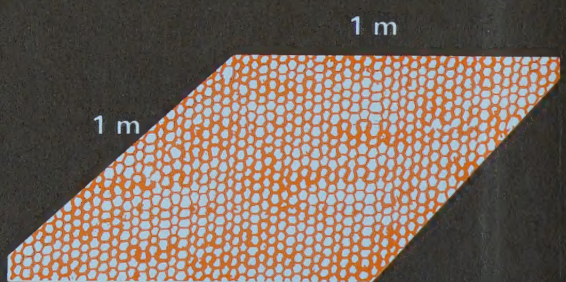
1 gram (1 g) is about the mass of a ping-pong ball

1 cubic centimetre ($1\ cm^3$) is smaller than a sugar cube

1 kilogram (1 kg) is about the mass of ten medium-sized apples

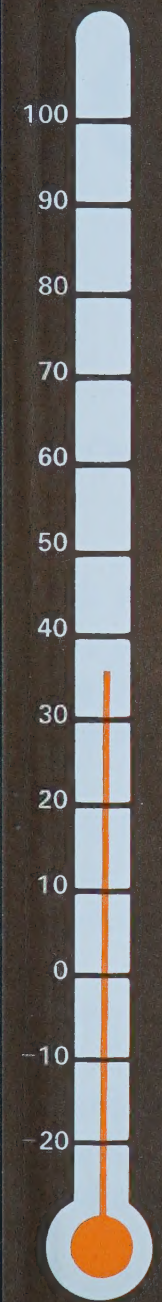


1 newton (1 N) is the gravitational force exerted on your hand by two golf balls

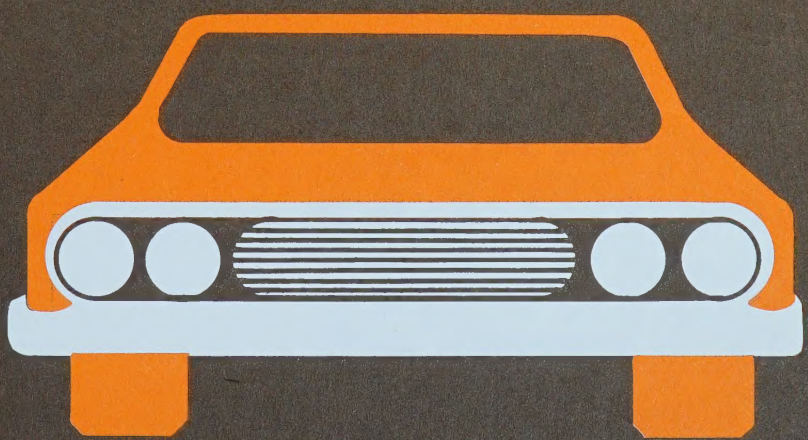


1 pascal (1 Pa) is the pressure produced on a square metre by 1 N (the two golf balls, ground up)

metric units



BICYCLE
mass about 10 kg
racing speed about 40 km/h



AUTOMOBILE
mass about 1950 kg
(or nearly 2 t)
highway speed about 100 km/h
gas tank holds about 90 L
tire pressure about 200 kPa

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Mathematics for a Modern World

Book 3, Second Edition

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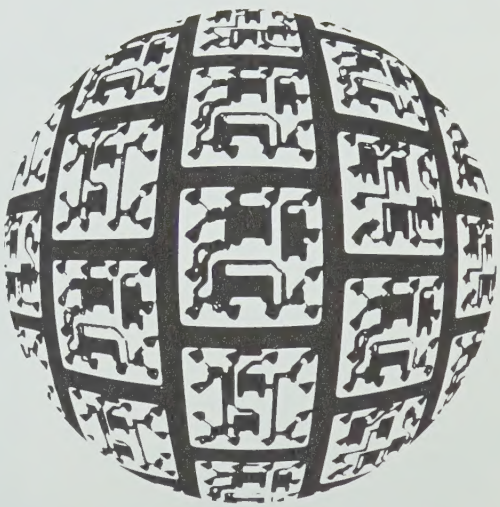

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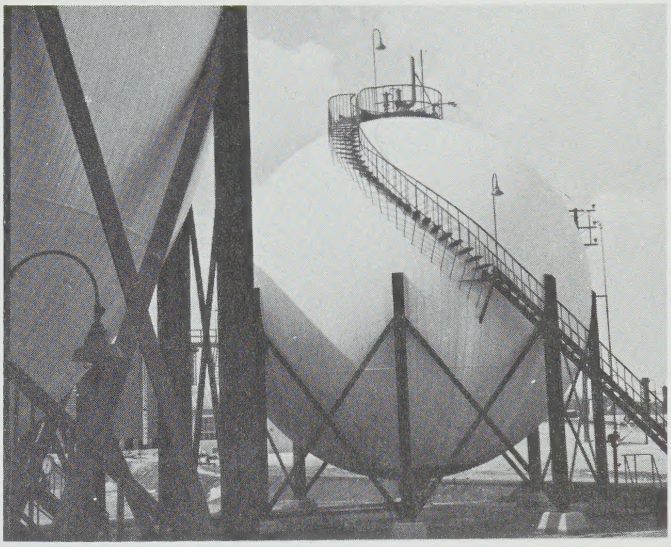
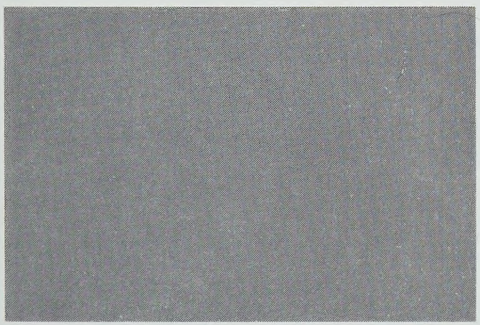
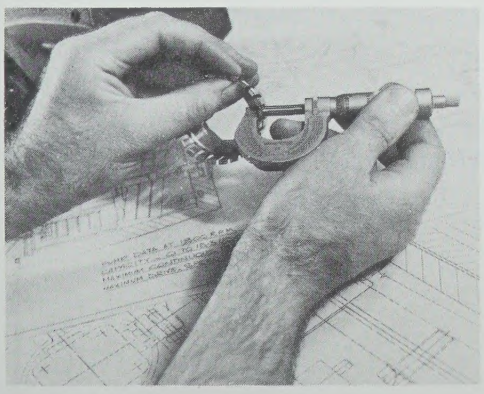
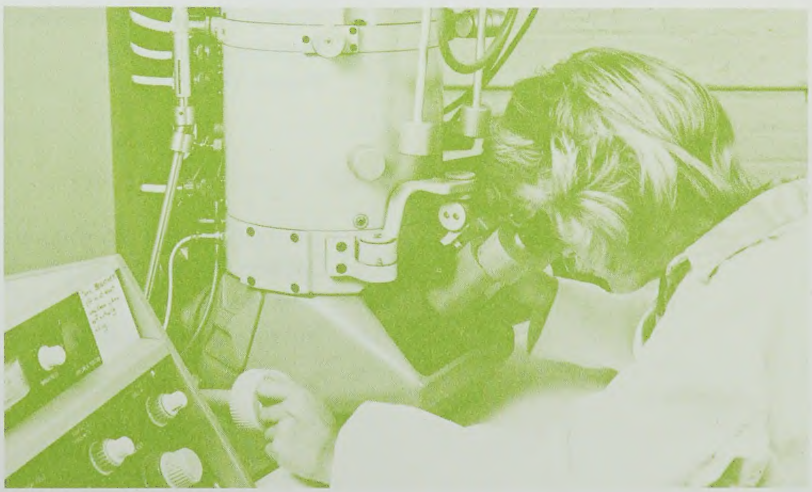
you learn
to understand
the world around you.









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Unit 1
Power Terms and Their Uses

In your study of the number systems, you traced the development through the sets of natural numbers (N), whole numbers (W), integers (I) and rationals (Q). A similar development can be followed in the study of numbers used as exponents in the formation of powers. In this chapter we will extend your knowledge of powers from natural number exponents to whole number and integral exponents, and conclude with an investigation of radical exponents.

How long would it take you to evaluate

$$\frac{44.87 \times 4775}{0.0471}?$$

Could you do it in *less than 2 min* without mechanical aids (slide rule or calculator)? Impossible? Not really. By the end of this unit you will be able to come back and time yourself on this problem, and win!

1.1 Get-Set Exercise

1. Find the sums.
   (*a*) $-9 + 5$
   (*b*) $5 + (-6)$
   (*c*) $-4 + (-8)$
   (*d*) $-8 + 17$
   (*e*) $-7 + (-9)$
   (*f*) $2 + (-2)$
   (*g*) $-3 + 3 - 3$
   (*h*) $-8 + 6 - 8 + 2$
   (*i*) $-7 + 4 - 3 + 7$
   (*j*) $8 + (-2) + (-8)$

2. Find the differences.
   (*a*) $-4 - 3$
   (*b*) $5 - 8$
   (*c*) $-9 - 2$
   (*d*) $-8 - (-4)$
   (*e*) $5 - (-3)$
   (*f*) $-3 - (-7)$
   (*g*) $-6 - (-2)$
   (*h*) $17 - (-4)$
   (*i*) $-16 - (-5)$
   (*j*) $-8 - (-9)$

$-8-(-4)$
$=-8+4$

3. Find the products.
   (*a*) $4 \times (-7)$
   (*b*) $-5 \times (-8)$
   (*c*) $-9 \times (-7)$
   (*d*) $-6 \times (-2)$
   (*e*) $-13 \times (-11)$
   (*f*) -6×0
   (*g*) -38×100
   (*h*) -10×26
   (*i*) -1×1
   (*j*) $0 \times (-8)$

4. Find the quotients.
   (*a*) $-18 \div (-3)$
   (*b*) $-12 \div 4$
   (*c*) $20 \div (-5)$
   (*d*) $-7 \div (-7)$
   (*e*) $0 \div (-7)$
   (*f*) $-28 \div (-4)$
   (*g*) $-36 \div 9$
   (*h*) $32 \div (-8)$
   (*i*) $-45 \div (-15)$
   (*j*) $128 \div (-2)$

$0 \div (-7)$
$=0$

5. Simplify.
   (*a*) $-10 \times (-2) \times (-4)$
   (*b*) $-12 \div 6 - (-4)$
   (*c*) $-8 - (-4) + (-8 - 4)$
   (*d*) $(-6 - 2) \div (-3 - 1)$
   (*e*) $(-8 + 2) \div (4 - 3)$
   (*f*) $-3[-3 + (-6)] - 5$
   (*g*) $-8 \times (-6) \div (-4) - 3 - (-9)$
   (*h*) $(6 + 8)(-5 - 4) - 32 \div (-4)$
   (*i*) $-5 \times 6 - (-3) - (-8)$
   (*j*) $(11 - 13)(2 - 8)$
   (*k*) $-5 + (-6) \times (-7) \div 3 + 4(12 - 16)$

Remember Order of Operations

$(11-13)(2-8)$
$=(-2)(-6)$
$=12$

(l) $\dfrac{18+6}{12\div(-2)}-\dfrac{14+6}{5+5}$

(m) $\dfrac{2\times7+6}{20\div2(-10)+(-2)}-\dfrac{8+7+(-21)}{-6\div(-1)}$

(n) $\dfrac{4\times3\times(-2)}{5\times(-2)\div5}+\dfrac{3+4\times5-2}{7-11-3}$

(o) $\dfrac{36\div(-9)}{-4+3}-\dfrac{(-21)\div(-7)}{6-3}$

(p) $\dfrac{1}{2}-\dfrac{1}{3}+\dfrac{2}{5}$

(q) $\dfrac{3}{4}-\dfrac{1}{12}+\dfrac{7}{24}$

(r) $\dfrac{3}{5}\times\dfrac{5}{7}\times\left(\dfrac{-2}{3}\right)$

(s) $\dfrac{5}{9}\div\left(-\dfrac{1}{3}\right)$

(t) $\dfrac{4}{7}\times\left(-\dfrac{7}{8}\right)\div\left(-\dfrac{2}{3}\right)$

$-\frac{1}{3}=\frac{-1}{3}$

$\frac{4}{7}\times\frac{-7}{8}\times\frac{3}{-2}$

6. Simplify.

(a) $\dfrac{(-5)(+3)}{(-15)}$

(b) $\dfrac{(-2)(+3)(-4)}{-6}$

(c) $\left(\dfrac{+28}{-4}\right)\times\left(\dfrac{+7}{-1}\right)$

(d) $\left(\dfrac{+12}{+3}\right)+\left(\dfrac{-27}{-9}\right)$

(e) $(-4)^2\times(-3)$

(f) $\left(\dfrac{+5}{-5}\right)^3$

(g) $\dfrac{(-8)-(+2)}{(+3)+(-5)}$

(h) $\dfrac{+3-(-12)}{-5}$

(i) $\left(\dfrac{-35}{-5}\right)-\left(\dfrac{-63}{+9}\right)$

(j) $(-12)(+2)-\left(\dfrac{+36}{-2}\right)$

7. If $x=-5$ and $y=2$, evaluate the following.

(a) $3x+2y$
(b) $7x-3y$
(c) $x+y$
(d) $2(x+y)^2$
(e) $-3(y-x)^2$
(f) $x-3y$
(g) $3x-5(-y)$
(h) x^2-3y^2
(i) $(x-y)^3$
(j) $(-2x+3y)^2$

8. Evaluate the following if $x=-3$ and $y=-4$.

(a) $8x+2(x-5)$
(b) $5y+2+3(2y+5)$
(c) $2x^2-3(y^2-2)$
(d) $7y-5+(-6y+2)$
(e) $3x-5(x+2)$
(f) $5-(2y+5)$
(g) $2(y+3)-(4-3y)$
(h) $5(3x^2-2)-(2x^2-3)$
(i) $3x-2(x+2y)-(-5-2y)$
(j) $3(x-y)^2+5x-3y^2$

$5-(2y+5)$
$=5-2y-5$
$=-2y$
$=-2(-4)$
$=8$

1.2 Natural Numbers as Powers

A POWER

a^x — x: EXPONENT; a: BASE

A 10^5 means $10 \times 10 \times 10 \times 10 \times 10$.
10^5 is the product of 5 tens.
10 is called the *base*.
5 is called the *exponent*.
10^5 is called a *power*.

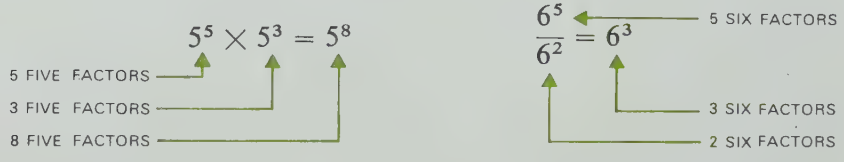
1. Express each of the following as a product.

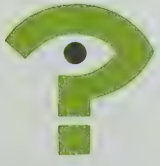
(*a*) 10^4	(*e*) 2^7	(*i*) 5^2
(*b*) 10^7	(*f*) 3^5	(*j*) 6^6
(*c*) a^3	(*g*) x^4	(*k*) c^4
(*d*) b^2	(*h*) z^7	(*l*) m^3

How many 10's in 10^n?

2. Express each of the following as a power.
   (*a*) $8 \times 8 \times 8$
   (*b*) $6 \times 6 \times 6 \times 6 \times 6$
   (*c*) $a \times a \times a \times a$
   (*d*) $y \times y \times y \times y \times y \times y$
   (*e*) 5×5
   (*f*) $7 \times 7 \times 7 \times 7 \times 7$
   (*g*) $9 \times 9 \times 9$
   (*h*) $x \times x \times x \times x \times x \times x$
   (*i*) $m \times m \times m \times m \times m \times m \times m$
   (*j*) $3 \times 3 \times 3$
   (*k*) 2×2
   (*l*) $z \times z \times z$

B Power terms with bases the same may be multiplied and divided very quickly. Here are two examples.





Can you see why? → $5^5 \times 5^3 = 5^{5+3} = 5^8$ and $\frac{6^5}{6^2} = 6^{5-2} = 6^3$

To multiply powers with like bases, add the exponents.

1. Find the product.

(*a*) $x^7 \times x^3$	(*d*) $a^3 \times a^7$	(*g*) $7^7 \times 7^6$
(*b*) $m^5 \times m^2$	(*e*) $t^3 \times t^{10}$	(*h*) $3^9 \times 3^3$
(*c*) $3^7 \times 3^4$	(*f*) $x^{15} \times x^7$	(*i*) $z^5 \times z^2$

2. Find the product.

(a) $3y^2 \times 2y^3$
(b) $4x^3 \times 3x^2$
(c) $5a^5 \times 3a^2$
(d) $7t^3 \times 3t^7$
(e) $12c^5 \times 4c^7$
(f) $-3m^3 \times 2m^2$
(g) $6m^5 \times 3m^7$
(h) $-4r^{14} \times 2r^8$
(i) $8c^3 \times 3c^8$
(j) $3p^3 \times 4p^4$

3. Find the quotient.

(a) $\frac{5^7}{5^3}$
(b) $\frac{a^5}{a^3}$
(c) $\frac{7^7}{7^3}$
(d) $\frac{9^{10}}{9^5}$
(e) $\frac{8^7}{8^2}$
(f) $\frac{x^9}{x^5}$
(g) $\frac{r^{14}}{r^{11}}$
(h) $\frac{t^9}{t^3}$
(i) $\frac{m^{23}}{m^{17}}$
(j) $\frac{x^{75}}{x^{65}}$
(k) $\frac{u^{43}}{u^{39}}$
(l) $\frac{b^{98}}{b^{89}}$

4. Find the quotient.

(a) $\frac{5y^4}{10y^2}$
(b) $22m^9 \div 11m^8$
(c) $49s^7 \div (-7s^6)$
(d) $25c^{14} \div 5c^9$
(e) $3x^7y^3 \div 6x^2y$
(f) $14r^7 \div 7r^5$
(g) $-18t^9 \div 9t^3$
(h) $4r^5s^7 \div 2r^2s^3$
(i) $27m^3n^5 \div 9m^2n^2$
(j) $-15x^3 \div 3x^2$
(k) $2r^9 \div 2r^7$
(l) $12a^{12} \div 4a^4$

5. Simplify.

(a) $10^{13} \div 10^5$
(b) $t^7 \times t^{10}$
(c) $x^{15} \div x^{13}$
(d) $(3c)^5 \div (3c)^2$
(e) $-6t^5 \div 3t^3$
(f) $-5a^4 \times 3a^7$

6. If $a^x = \underbrace{a \times a \times a \times a \times \cdots \times a}_{x \text{ factors}}$, what kind of number must x be?

7. If $a^4 = a \times a \times a \times a$
$a^3 = a \times a \times a$
$a^2 = a \times a$
$a^1 = \square$

Remember?
$3y^2$ means $3 \times y \times y$
and
$2y^3$ means $2 \times y \times y \times y$.

To divide powers with like bases, subtract the exponents.

$\frac{5y^4}{10y^2} = \frac{\overset{1}{\cancel{5}} \times \overset{1}{\cancel{y}} \times \overset{1}{\cancel{y}} \times y \times y}{\underset{1}{\cancel{10}} \times \underset{1}{\cancel{y}} \times \underset{1}{\cancel{y}}}$
$= \frac{y^2}{2}$

C Sometimes the power terms involve brackets, and we then have powers of powers.

$(t^2)^3$ means $t^2 \times t^2 \times t^2$
$= t^6$

$(a^2b)^4$ means $a^2b \times a^2b \times a^2b \times a^2b$
$= a^8b^4$

$\left(\frac{m}{n}\right)^3$ means $\frac{m}{n} \times \frac{m}{n} \times \frac{m}{n}$
$= \frac{m^3}{n^3}$

$(t^2)^3$ is the third power of t squared. A power of a power!

Can you see a fast way to find the exponent?
$(a^5)^3 = a^{15}$ and
$(m^4)^5 = m^{20}$

1. Read the following.
   (a) $(m^7)^2$
   (b) $(mn)^5$
   (c) $\left(\frac{a^2}{b}\right)^3$
   (d) $(3^3)^3$
   (e) $(-3^2)^4$
   (f) $(x^2y)^4$
   (g) $\left(\frac{-3e^2}{f}\right)^5$

2. Remove the brackets.
   (a) $(e^4)^2$
   (b) $(g^7)^3$
   (c) $(a^3b^2)^2$
   (d) $(m^5n^2)^3$
   (e) $(cd^3)^4$
   (f) $\left(\frac{a}{c}\right)^5$
   (g) $\left(\frac{a^3}{b^2}\right)^7$
   (h) $\left(\frac{x^3y^2}{m^3}\right)^5$

Careful! Some of these are sneaky.

Exercise 1.2

Discussion

1. Name the base and the exponent.
   (a) 7^{43}
   (b) a^6
   (c) x^7
   (d) $3m^3$
   (e) $4r^{14}$
   (f) $-2a^{14}$

2. Express as a product.
   (a) r^3
   (b) s^7
   (c) 2^3
   (d) 3^7
   (e) $(-7)^3$
   (f) $(-5)^4$
   (g) m^5
   (h) b^6
   (i) $(-t)^4$

3. Express as a power.

(a) $m \times m \times m \times m \times m \times m \times m$
(b) $-2 \times -2 \times -2 \times -2 \times -2$
(c) $c \times c \times c \times c \times c \times c \times c$
(d) $4 \times 4 \times 4$
(e) $5^2 \times 5^2 \times 5^2$
(f) $7^5 \times 7^2 \times 7^1$

4. Simplify.

(a) $10^2 \times 10^3$
(b) $2x^3 \times 3x^5$
(c) $x^2 \times x^5$
(d) $c^{18} \div c^9$
(e) $m^7 \div m^3$
(f) $4p^{30} \div 2p^{15}$
(g) $2^7 \div 2^2$
(h) $10^{10} \times 10^{10}$

5. Remove the brackets.

(a) $(a^3)^2$
(b) $\left(\frac{2}{3}\right)^3$
(c) $\left(\frac{m^3}{n^2}\right)^4$
(d) $\left(\frac{3^2}{5}\right)^2$
(e) $(r^4)^2$
(f) $(-c^2)^3$
(g) $\left(\frac{a}{b^3}\right)^4$
(h) $((-c)^2)^3$
(i) $\left(\frac{x^2}{-y^3}\right)^3$

Written Solutions

6. Simplify..

(a) $3y^2 \times 2y^3$
(b) $2^{10} \times 2^{12} \div 2^{18}$
(c) $3^{15} \times 3^{25} \div 3^{10} \div 3^{27}$
(d) $\left(\frac{3}{5}\right)^5 \div \left(\frac{3}{5}\right)^3$
(e) $3^2 \times 2^3$
(f) $2r^3 \times 5r^5$
(g) $m^9 \div m^7$
(h) $\left(\frac{2}{3}\right) \times \left(\frac{2}{3}\right)^2$
(i) $(-2)^4 \div 2^5$
(j) $3m^7 \div 15m^6$
(k) $(-2x)^5 \times (-2x)^3$
(l) $3y^5 \div 6y^4$

7. Remove the brackets and simplify.

(a) $(x^3)^3$
(b) $\left(\frac{2}{3}\right)^4 \div \left(\frac{2}{3}\right)^2$
(c) $\left(\frac{-m^3}{n^2}\right)^3 \times \left(\frac{-m^2}{n^3}\right)^2$
(d) $\left(\frac{m}{n^2}\right)^3$
(e) $\left(-\frac{x}{y}\right)^5 \div \left(-\frac{x}{y}\right)^3$
(f) $(x^3)^3 \times (y^2)^3 \div (x^2)^5$
(g) $\left(\frac{3^2}{5}\right)^3$
(h) $(2^4)^3 \div (2^5)^2$

3 (e) How many 5's as factors?

Be careful in 6 (k) and (i)!

6(a) $6y^5$
(c) 27
(e) 72
(i) $\frac{1}{2}$
(l) $\frac{y}{2}$

7(a) x^9
(d) $\frac{m^3}{n^6}$
(g) $\frac{3^6}{5^3}$
(h) 2^2

8(b) $12x^5y^3$
(d) 6×10^{12}
(e) 3^3
(g) $6x^{12}$

8. Simplify the following.

(a) $2^3 \times (-2)^4$
(b) $3x^2y \times 4x^3y^2$
(c) $(5c^3d^5)^4 \div (5c^5d^7)^3$
(d) $3 \times 10^5 \times 2 \times 10^7$
(e) $((-3)^2)^3 \div 3^3$
(f) $(2x^3y^2)^3 \div (-2xy^2)^2$
(g) $3x^5 \times 2x^7$
(h) $(4.3 \times 10^4)(2.0 \times 10^5)$

Questions 9 and 10 are special. Memorize the results you get.

9. Complete the following statements.

(a) $a^x \times a^y =$ ▤ LEM
(Law of Exponents for Multiplication)

(b) $a^x \div a^y =$ ▤ LED
(Law of Exponents for Division)

10. Remove the brackets.

(a) $(a^x)^y =$ ▤ LEP
(Law of Exponents for Powers)

(b) $\left(\frac{a}{b}\right)^y =$ ▤ LEPQ
(Law of Exponents for Power of a Quotient)

(c) $(ab)^x =$ ▤ LEPP
(Law of Exponents for Power of a Product)

1.3 Zero and Negative Numbers as Powers

Except zero, of course.

A Any number divided by itself is 1.

$$\therefore \frac{10^5}{10^5} = \frac{10 \times 10 \times 10 \times 10 \times 10}{10 \times 10 \times 10 \times 10 \times 10}$$
$$= 1$$
$$a^7 \div a^7 = 1$$
$$c^3 \div c^3 = 1$$

Do this division using the division rule for powers with like bases.

$$\frac{10^5}{10^5} = 10^{5-5} = 10^0$$
$$a^7 \div a^7 = a^{7-7} = a^0$$
$$c^3 \div c^3 = c^{3-3} = c^0$$

(Any non-zero number)^0=1

Therefore if the division rule is to hold, $10^0 = 1$ and $a^0 = 1$ and $c^0 = 1$, even if it doesn't look correct.

If you're not convinced, try this explanation.

$10^3 = 1000$ Divide by 10.
$10^2 = 100$
$10^1 = 10$
$10^0 = ?$

Or this one.

$$10^3 \times 1 = 10^3$$
$$\text{Also } 10^3 \times 10^0 = 10^{3+0} = 10^3$$
$$\therefore 10^0 = 1$$

Convinced? Then try division!

$$\frac{10^4}{10^0} = 10^{4-0} = 10^4$$
$$\text{But } \frac{10^4}{1} = 10^4$$
$$\therefore 10^0 = 1$$

. . . Almost!

Now I'll buy it!

B Study this example.

$$c^3 \div c^5 = \frac{c \times c \times c}{c \times c \times c \times c \times c} = \frac{1}{c^2}$$

The division rule LED gives

$c^3 \div c^5 = c^{3-5} = c^{-2}$ which means that

$$c^{-2} = \frac{1}{c^2}$$

In order to use the division rule for like bases in all situations, we must agree on two new definitions.

It's an agreement among mathematicians.

$$x^0 = 1 \text{ and } x^{-a} = \frac{1}{x^a}$$

1. Rewrite using positive exponents.

(a) a^{-3} (d) $\frac{1}{m^{-3}}$ (g) r^{-2} (j) 5^{-2}

(b) x^{-5} (e) $\frac{1}{t^{-2}}$ (h) m^{-3} (k) $\frac{1}{7^{-4}}$

(c) m^{-2} (f) $\frac{1}{r^{-4}}$ (i) 6^{-3} (l) $\frac{1}{10^{-3}}$

2. Rewrite using negative exponents.

(*a*) $\frac{1}{5^3}$ (*c*) $\frac{1}{x^3}$ (*e*) $\frac{1}{m^2n^3}$ (*g*) $\frac{1}{x^3y^2}$

(*b*) $\frac{1}{7^4}$ (*d*) $\frac{1}{y^5}$ (*f*) $\frac{1}{a^5b^2}$ (*h*) $\frac{1}{10^3}$

C The laws of exponents still hold for exponents that are negative as this example shows.

y^{-5} means $\frac{1}{y^5}$ and y^{-3} means $\frac{1}{y^3}$

$$\text{Therefore } y^{-5} \times y^{-3} = \frac{1}{y^5} \times \frac{1}{y^3}$$
$$= \frac{1}{y^8} = y^{-8}$$
$$\text{Also } y^{-5} \times y^{-3} = y^{-8} \qquad \text{Using LEM.}$$

Does LED work for negative exponents?

$$x^{-4} \div x^{-7} = \frac{1}{x^4} \div \frac{1}{x^7}$$
$$= \frac{1}{x^4} \times \frac{x^7}{1}$$
$$= x^{7-4}$$
$$= x^3$$
$$\text{But } x^{-4} \div x^{-7} = x^{-4-(-7)}$$
$$= x^{-4+7}$$
$$= x^3 \qquad \text{Using LED.}$$

I guess it does!

Finally let's try powers of powers.

$$(c^{-3})^2 = c^{-3} \times c^{-3}$$
$$= c^{-6} \qquad \text{Using LEM.}$$
$$\text{But } (c^{-3})^2 = c^{(-3)2}$$
$$= c^{-6} \qquad \text{Using LEP.}$$

1. Find the product.

(*a*) $x^{-2} \times x^{-4}$ (*d*) $7^{-2} \times 7^{-7}$ (*g*) $m^{-5} \times m^{-7}$

(*b*) $a^{-3} \times a^{-2}$ (*e*) $3^{-3} \times 3^{-14}$ (*h*) $9^{-9} \times 9^{-3}$

(*c*) $b^{-5} \times b^{-7}$ (*f*) $y^{-4} \times y^{-6}$ (*i*) $4^{-2} \times 4^{-3}$

2. Find the product.

(a) $x^{-2} \times x^5$	(d) $r^{-3} \times r^7$	(g) $5^{-2} \times 5^7$
(b) $x^5 \times x^{-4}$	(e) $s^{-2} \times s^7$	(h) $7^{-9} \times 7^2$
(c) $a^{-7} \times a^4$	(f) $m^{-7} \times m^2$	(i) $9^{-5} \times 9^3$

3. Find the quotient.

(a) $x^7 \div x^9$	(d) $m^2 \div m^5$	(g) $15^{-7} \div 15^{-9}$
(b) $5^3 \div 5^5$	(e) $m^{-3} \div m^2$	(h) $3^7 \div 3^{-4}$
(c) $7^{-4} \div 7^{-8}$	(f) $a^{-3} \div a^{-4}$	(i) $4^{-2} \div 4^{-3}$

4. Remove the brackets and simplify.

(a) $(m^{-4})^2$	(d) $(c^{-2})^{-2}$	(g) $(u^{-7})^5$
(b) $(t^{-3})^3$	(e) $(f^{-3})^{-5}$	(h) $(x^{-1})^{-2}$
(c) $(d^{-5})^5$	(f) $(x^{-2})^{-4}$	(i) $(y^{-3})^2$

Exercise 1.3

Discussion

1. State the product.

(a) $a^2 \times a^3$	(c) $m^{-3} \times m^4$	(e) $t^2 \times t^{-3}$
(b) $y^{-5} \times y^{-3}$	(d) $s^3 \times s^{-3}$	(f) $r^{-2} \times r^{-12}$

2. State the quotient.

(a) $x^3 \div x^2$	(d) $m^4 \div m^5$	(g) $m^{-2} \div m^3$
(b) $y^4 \div y^{-2}$	(e) $t^{-3} \div t^{-4}$	(h) $r^{-5} \div r^{-2}$
(c) $c^{-15} \div c^{-16}$	(f) $f^{-3} \times f^{-2}$	(i) $x^{-3} \div x^{-3}$

3. Simplify.

(a) $5^2 \div 5^0$

(b) $2^3 \times 2^4 \div 2^5 \times 3^0$

(c) $15^{17} \times 15^{-10} \div 15^7$

(d) $3^5 \times 3^{14} \times 3^{-10} \times 3 \times 3^{-8}$

4. Rewrite with positive exponents.

(a) 2^{-3}	(d) a^{-4}	(g) $\frac{1}{m^{-4}}$
(b) $\frac{1}{5^{-2}}$	(e) $\frac{1}{z^{-3}}$	(h) a^{-7}
(c) 3^{-7}	(f) r^{-7}	(i) $\frac{1}{b^{-13}}$

Remember the Laws

LEM

$a^x \times a^y = a^{x+y}$

LED

$a^x \div a^y = a^{x-y}$

LEP

$(a^x)^y = a^{xy}$

Written Solutions

5. Simplify.
   (*a*) $10^5 \times 10^2 \times 10^{-7} \times 10^3$
   (*b*) $5^3 \times 5^2 \times 3^3 \times 5^{-5} \times 3^{-1}$
   (*c*) $5x^{-3} \times \frac{x^5}{15} \times 27x^{-3}$
   (*d*) $10^7 \div 10^5 \times 10^{-2}$
   (*e*) $3^4 \div 3^{-4} \times 3^{-7}$
   (*f*) $2^3 \times 2^4 \times 2^{-5} \times 2^{-3}$
   (*g*) $(-7)^3 \times (-7)^{-1} \times (-7)^{-5} \times (-7)$
   (*h*) $(-2)^{-3} \times (-2)^4 \times (-2)^5 \div (-2)^3$
   (*i*) $\left(\frac{2}{3}\right)^4 \times \left(\frac{2}{3}\right)^{-2} \div \left(\frac{2}{3}\right)^3$

6. Find the products of $5^3 \times 5^0$ and $5^3 \times 1$ and use your results to show that $5^0 = 1$.

7. Simplify.
   (*a*) $\frac{3^2 \times 3^4}{3^5}$
   (*b*) $\frac{x^5 \times x^7 \times x^{-3}}{x^9}$
   (*c*) $\frac{r^{-7} \times r^{-13}}{r^{-15}}$
   (*d*) $\frac{5^3 \times 5^5 \times 5^2}{5^3 \times 5^7}$
   (*e*) $\frac{a^{-5} \times a^4 \times a^3}{a^2}$
   (*f*) $\frac{(-3t)^2 \times (-3t)^{-3} \times (-3t)^7}{(-3t)^3 \times (-3t)^2}$
   (*g*) $\frac{2^3 \times 5^2 \times 2^4}{2^5 \times 2^2}$
   (*h*) $\frac{3x^3 \times 4x^2}{6x^{-5}}$

8. Simplify.
   (*a*) $\frac{2.8 \times 10^4}{1.4 \times 10^3}$
   (*b*) $\frac{3 \times 10^2 \times 4 \times 10^{-5}}{6 \times 10^{-3}}$
   (*c*) $\frac{4 \times 10^{-3} \times 9 \times 10^9}{12 \times 10^5}$
   (*d*) $\frac{6 \times 10^5 \times 5 \times 10^{-3}}{15 \times 10^2}$
   (*e*) $\frac{15 \times 10^3 \times 12 \times 10^5 \times 6 \times 10^{-4}}{30 \times 10^7 \times 18 \times 10^{-5}}$

5(a) 10^3
(c) $9x^{-1}$
(e) 3
(f) 2^{-1}
(i) $(\frac{2}{3})^{-1}$

7(b) 1
(d) 1
(f) $-3t$
(h) $2x^{10}$

8(b) 2
(d) 2

9(a) 3600
(e) 0.0034

9. Simplify.

(a) $\dfrac{7200 \times 0.18}{0.36}$ (d) $\dfrac{18\,600 \times 150}{0.031}$

(b) $\dfrac{580 \times 0.3}{17\,400}$ (e) $\dfrac{0.000\,136 \times 0.02}{0.0008}$

(c) $\dfrac{140 \times 0.3}{4200}$ (f) $\dfrac{20\,000 \times 0.09}{180}$

Hint for 9(a)
$7200 = 7.2 \times 10^3$
$0.18 = 1.8 \times 10^{-1}$
$0.36 = 3.6 \times 10^{-1}$

1.4 Rational Exponents

RATIONAL NUMBERS ⟷ FRACTIONS

A Taking Roots.

$\sqrt{9} = 3$ *
$\sqrt{25} = 5$ } Why?
$\sqrt{64} = 8$

$\sqrt{9} = \sqrt{3^2} = 3$
$\sqrt{25} = \sqrt{5^2} = 5$
$\sqrt{64} = \sqrt{2^6} = 2^3 = 8$

* $3 \times 3 = 9$
$\therefore \sqrt{9} = 3$

What happens to the exponent when you take the square root?

$$\sqrt{16} = \sqrt{2^4} = 2^2 = 4$$

Does your theory still work?

1. Try these.

(a) $\sqrt{3^6}$ (c) $\sqrt{5^4}$ (e) $\sqrt{7^6}$ (g) $\sqrt{6^4}$

(b) $\sqrt{625}$ (d) $\sqrt{81}$ (f) $\sqrt{11^4}$ (h) $\sqrt{13^4}$

$\sqrt[3]{8} = 2$ because $\sqrt[3]{8} = \sqrt{2^3} = 2$ *

$\sqrt[3]{27} = 3$ because $\sqrt[3]{27} = \sqrt[3]{3^3} = 3$

* $2 \times 2 \times 2 = 8$
$\therefore \sqrt[3]{8} = 2$

Can you evaluate $\sqrt[3]{125}$? $\sqrt[3]{64}$?

What do you do to the exponent of a power in order to take the cube root?

What do you think you would do for the fourth root? The fifth root?

What is your rule?

Rule:
If $k = p^n$
then $\sqrt[n]{k} = p$.

Simplify.

(a) $\sqrt[3]{6^3}$ (c) $\sqrt[5]{2^5}$ (e) $\sqrt[4]{3^4}$ (g) $\sqrt{81}$

(b) $\sqrt[6]{64}$ (d) $\sqrt[3]{1000}$ (f) $\sqrt[5]{2^{10}}$ (h) $\sqrt[12]{10^{36}}$

B We can extend our knowledge of exponents now by considering the following examples.

$$x^4 = x^2 \times x^2 \qquad \therefore \sqrt{x^4} = x^2$$
$$x^2 = x^1 \times x^1 \qquad \therefore \sqrt{x^2} = x$$
$$x^1 = x^{\frac{1}{2}} \times x^{\frac{1}{2}} \qquad \therefore \sqrt{x^1} = x^{\frac{1}{2}}$$

Similarly,

$$x^6 = x^2 \times x^2 \times x^2 \qquad \therefore \sqrt[3]{x^6} = x^2$$
$$x^3 = x \times x \times x \qquad \therefore \sqrt[3]{x^3} = x$$
$$x^1 = x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} \qquad \therefore \sqrt[3]{x^1} = x^{\frac{1}{3}}$$

Good news!

Consequently the rule developed for taking roots of a power can be used *all* the time.

1. Rewrite with rational exponents.

(*a*) $\sqrt[3]{x^3}$	(*f*) $\sqrt[4]{y}$	(*k*) $\sqrt[7]{x^{14}}$
(*b*) $\sqrt[5]{m^{15}}$	(*g*) $\sqrt[5]{z}$	(*l*) $\sqrt{5^2}$
(*c*) $\sqrt[4]{8^8}$	(*h*) $\sqrt[7]{2}$	(*m*) $\sqrt[4]{a^{12}}$
(*d*) $\sqrt[3]{5^6}$	(*i*) $\sqrt[9]{3}$	(*n*) $\sqrt[3]{15^6}$
(*e*) $\sqrt[3]{x}$	(*j*) $\sqrt[5]{m}$	(*o*) $\sqrt{7}$

$x^{\frac{a}{b}} = \sqrt[b]{x^a}$

C We have seen that $\sqrt[3]{x} = x^{\frac{1}{3}}$.
If we use the same rule on $\sqrt[3]{x^2}$, we will get

$$\sqrt[3]{x^2} = (x^2)^{\frac{1}{3}}$$
$$= x^{\frac{2}{3}} \qquad \text{LEP}$$

Similarly, $\sqrt[5]{7^3} = (7^3)^{\frac{1}{5}}$
$$= 7^{\frac{3}{5}}$$

Do you see why $\sqrt[6]{a^5} = a^{\frac{5}{6}}$
and $\sqrt[4]{7^3} = 7^{\frac{3}{4}}$?

1. Rewrite with rational exponents.

(*a*) $\sqrt[3]{x^2}$	(*d*) $\sqrt[9]{5^3}$	(*g*) $\sqrt[5]{m^3}$
(*b*) $\sqrt[5]{a^4}$	(*e*) $\sqrt[3]{2^2}$	(*h*) $\sqrt[5]{y^4}$
(*c*) $\sqrt[7]{y^2}$	(*f*) $\sqrt[7]{4^3}$	(*i*) $\sqrt[3]{z^3}$

Exercise 1.4

Discussion

1. Read these with rational exponents.

(a) $\sqrt[5]{y}$	(c) $\sqrt[7]{x}$	(e) $\sqrt[5]{125}$	(g) $\sqrt[3]{16}$
(b) $\sqrt[10]{20}$	(d) $\sqrt[7]{m}$	(f) $\sqrt[12]{t}$	(h) $\sqrt[9]{3}$

2. Express using a radical sign.

(a) $z^{\frac{1}{3}}$	(c) $m^{\frac{1}{5}}$	(e) $f^{\frac{1}{4}}$	(g) $x^{\frac{1}{2}}$
(b) $b^{\frac{1}{2}}$	(d) $a^{\frac{1}{6}}$	(f) $d^{\frac{1}{10}}$	(h) $y^{\frac{1}{8}}$

3. Express using rational exponents.

(a) $\sqrt[7]{a}$	(c) $\sqrt[3]{x}$	(e) $\sqrt[3]{t}$	(g) $\sqrt{c}$
(b) $\sqrt[5]{m}$	(d) $\sqrt{r}$	(f) $\sqrt[10]{b}$	(h) $\sqrt[12]{e}$

4. Read with rational exponents.

(a) $\sqrt[7]{z^2}$	(c) $\sqrt[5]{a^3}$	(e) $\sqrt[4]{c^3}$	(g) $\sqrt[5]{4}$
(b) $\sqrt[8]{7}$	(d) $\sqrt[9]{2^4}$	(f) $\sqrt[10]{5^3}$	(h) $\sqrt[3]{9}$

5. Read using a radical sign.

(a) $4^{\frac{3}{5}}$	(c) $7^{\frac{2}{3}}$	(e) $2^{\frac{3}{4}}$	(g) $x^{\frac{5}{2}}$
(b) $4^{\frac{1}{9}}$	(d) $8^{\frac{3}{7}}$	(f) $10^{\frac{2}{5}}$	(h) $m^{\frac{7}{4}}$

Written Solutions

6. Simplify.

(a) $32^{\frac{1}{5}}$	(i) $9^{0.5}$	(q) $25^{\frac{3}{2}}$
(b) $216^{\frac{1}{3}}$	(j) $243^{0.2}$	(r) $64^{\frac{2}{6}}$
(c) $16^{0.25}$	(k) $16^{\frac{3}{2}}$	(s) $64^{\frac{1}{3}}$
(d) $81^{0.25}$	(l) $81^{\frac{3}{4}}$	(t) $-1^{\frac{1}{7}}$
(e) $8^{\frac{2}{3}}$	(m) $625^{\frac{1}{4}}$	(u) $49^{0.5}$
(f) $100^{\frac{3}{2}}$	(n) $8^{\frac{1}{3}}$	(v) $121^{0.5}$
(g) $27^{\frac{1}{3}}$	(o) $36^{0.5}$	(w) $64^{\frac{4}{3}}$
(h) $343^{\frac{1}{3}}$	(p) $-1^{0.2}$	(x) $125^{\frac{2}{3}}$

Rational Exponents

$\sqrt[4]{a} = a^{\frac{1}{4}}$

$\sqrt[4]{a^2} = a^{\frac{2}{4}}$

6(a) 2
(d) 3
(g) 3
(j) 3
(m) 5
(p) −1
(r) 4
(u) 7
(x) 25

1.5 Standard Form and Its Uses

Writing numbers in standard form involves powers of 10.

$$4830 = 4.83 \times 10^3$$
$$0.0432 = 4.32 \times 10^{-2}$$

These numbers must be between 1 and 10.

Scientists use standard form when working with very large and very small numbers.

The time it takes light to cross a room is about 10^{-7} s.

Engineers use standard form to estimate an approximate answer before using their slide rules.

$$\frac{8470 \times 0.0047}{0.02} = \frac{8.47 \times 10^3 \times 4.7 \times 10^{-3}}{2.0 \times 10^{-2}}$$
$$= \frac{8.47 \times 4.7 \times 10^2}{2.0}$$
$$\doteq \frac{\overset{4}{\cancel{8}} \times 5}{\underset{1}{\cancel{2}}} \times 10^2$$
$$= 2000$$

$\doteq$ means "is approximately equal to."

To write any decimal number in standard form, the decimal point is relocated so that only one non-zero digit is to the left.

Standard position
There is one non-zero digit to the left.

Thus for 8394. you relocate the decimal point
8.394 and compensate by a power of 10.
Therefore $8394 = 8.394 \times 10^{3}$

The decimal was moved three places.

Example 1

Write 16 400. in standard form.

Solution

$16\,400. = 1.64 \times 10^{4}$

DECIMAL IS 4 RIGHT OF STANDARD POSITION
STANDARD POSITION

Example 2

Write 0.000 438 in standard form.

Do you see a short cut?

Solution

$0.000\,438 = 4.38 \times 10^{-4}$

STANDARD POSITION
DECIMAL IS 4 LEFT OF STANDARD POSITION

Example 3

Write 843 000 in standard form.

Solution

$843\,000. = 8.43 \times 10^{5}$

5 RIGHT

Example 4

Write 0.000 001 63 in standard form.

Solution

$0.000\,001\,63 = 1.63 \times 10^{-6}$

6 LEFT

Exercise 1.5

Discussion

1. State the standard position for the decimal point.

(*a*) 1489	(*d*) 0.004 32	(*g*) 13.47	(*j*) 9761.2
(*b*) 13 742	(*e*) 0.000 005 79	(*h*) 1997.2	(*k*) 18 430
(*c*) 12.3	(*f*) 0.0147	(*i*) 745.2	(*l*) 656.4

2. State the power of 10 necessary to express the following numbers in standard form.

(*a*) 4376	(*d*) 0.000 004 01	(*g*) 0.000 481	(*j*) 1.002
(*b*) 0.004 97	(*e*) 16 735	(*h*) 379.1	(*k*) 472.1
(*c*) 0.1279	(*f*) 7.23	(*i*) 0.0423	(*l*) 0.000 30

3. Read the following without the power of 10.

(*a*) 1.72×10^{4}	(*d*) 5.43×10^{-3}	(*g*) 4.93×10^{-7}
(*b*) 4.79×10^{7}	(*e*) 7.91×10^{-4}	(*h*) 3.65×10^{4}
(*c*) 9.21×10^{2}	(*f*) 1.67×10^{5}	(*i*) 7.29×10^{-3}

4. State in standard form.

(*a*) 1083	(*c*) 1 096 000	(*e*) 0.000 89
(*b*) 0.943	(*d*) 64 300	(*f*) 0.0417

5(a) 1.3×10^3
(d) 2.5×10^4
(e) 1.1×10
(h) 1.6×10^{-12}

Written Solutions

5. Convert to standard form and estimate the answer.

(a) $\dfrac{847 \times 462}{271}$ (e) $\dfrac{406\,000 \times 0.023}{672}$

(b) $\dfrac{1\,046\,000 \times 0.003\,21}{597\,000}$ (f) $\dfrac{0.876 \times 574\,000}{0.985}$

(c) $\dfrac{64\,000 \times 8405}{0.000\,16}$ (g) $\dfrac{0.000\,48 \times 0.000\,000\,2}{0.000\,000\,000\,096}$

(d) $\dfrac{436 \times 0.0472}{0.000\,81}$ (h) $\dfrac{0.000\,793 \times 0.003\,73}{1\,946\,000}$

1.6 Further Uses of Standard Form

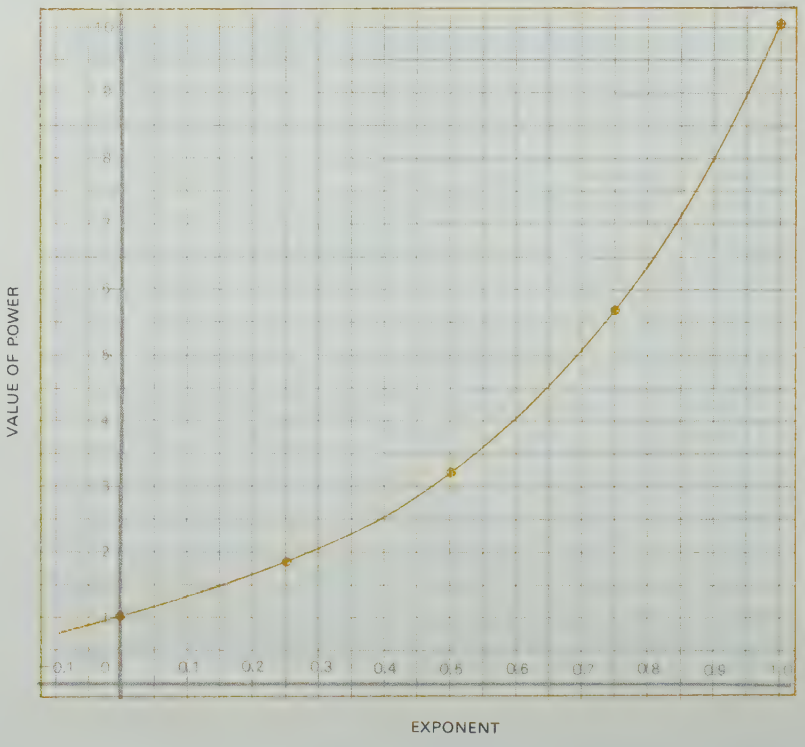
A We know that $10^0 = 1$ and $10^1 = 10$. Therefore $10^{\frac{1}{2}}$ is between 1 and 10. From a slide rule $10^{\frac{1}{2}} \doteq 3.16$

$10^{\frac{1}{4}} \doteq 1.78$

$10^{\frac{3}{4}} \doteq 5.62$

These results have been plotted in the diagram. From the graph it is possible to estimate that

$10^{0.1} \doteq 1.3$ and $10^{0.2} \doteq 1.6$



1. Estimate from the graph.

(*a*)	$10^{0.3}$	(*d*)	$10^{0.4}$	(*g*)	$10^{0.6}$	(*j*)	$10^{0.7}$
(*b*)	$10^{0.8}$	(*e*)	$10^{0.9}$	(*h*)	$10^{\frac{2}{3}}$	(*k*)	$10^{\frac{5}{6}}$
(*c*)	$10^{\frac{1}{3}}$	(*f*)	$10^{\frac{1}{7}}$	(*i*)	$10^{\frac{7}{8}}$	(*l*)	$10^{\frac{7}{9}}$

2. Estimate the value for the exponent x.

(*a*)	$10^x = 2$	(*d*)	$10^x = 3$	(*g*)	$10^x = 4$
(*b*)	$10^x = 5$	(*e*)	$10^x = 6$	(*h*)	$10^x = 7$
(*c*)	$10^x = 8$	(*f*)	$10^x = 9$	(*i*)	$10^x = 0.45$

B Tables have been prepared that give more accurate values than those obtained from the graph. An excerpt of such a table is shown here.

Exp.	0	1	2	3	4	5	6	7	8	9
0.65	4.467	4.477	4.487	4.498	4.508	4.519	4.529	4.539	4.550	4.560
0.66	4.571	4.581	4.592	4.603	4.613	4.624	4.634	4.645	4.656	4.667
0.67	4.677	4.688	4.699	4.710	4.721	4.732	4.742	4.753	4.764	4.775

The left-hand column and the top row give the first three decimal places for the exponent. The table itself gives the value of the power. Note that all numbers in the body of the table are in standard form.

Example 1

$$0.046\,13 = 4.613 \times 10^{-2}$$
$$= 10^{0.664} \times 10^{-2}$$

Find in the body of the table.

From the left column and the top row.

This method can be used to solve problems where more accuracy is required than a slide rule can give.

Example 2

$$4477. \times 0.004\,753 = 4.477 \times 10^3 \times 4.753 \times 10^{-3}$$
$$\doteq 10^{0.651} \times 10^3 \times 10^{0.677} \times 10^{-3}$$
$$= 10^{0.651} \times 10^{0.677}$$
$$= 10^{1.328}$$
$$= 10^{0.328} \times 10^1$$
$$\doteq 2.128 \times 10^1$$
$$= 21.28$$

STEPS

1. WRITE NUMBERS IN STANDARD FORM.
2. USE TABLE TO GET POWER.
3. ADD EXPONENTS.
4. USE TABLE TO CONVERT POWER TO A NUMBER BETWEEN 1 AND 10.
5. WRITE WITHOUT A POWER OF 10.

Example 3

$$\begin{aligned} 3243 \div 1091 &= 3.243 \times 10^3 \div (1.091 \times 10^3) \\ &\doteq \frac{10^{0.511} \times 10^3}{10^{0.038} \times 10^3} \\ &= 10^{0.511 - 0.038} \\ &= 10^{0.473} \\ &\doteq 2.972 \end{aligned}$$

Exercise 1.6

Written Solutions

1. Use tables to convert to powers of 10.
   (*a*) 1.081
   (*b*) 5.333
   (*c*) 3.459
   (*d*) 7.079
   (*e*) 8.974
   (*f*) 1.991
   (*g*) 6.067
   (*h*) 2.911

2. Convert to powers of 10 and solve.
   (*a*) 2.46×14.72
   (*b*) $0.000\,417 \times 984$
   (*c*) $0.000\,683\,9 \times 0.007\,43$
   (*d*) $548\,300 \div 272.9$
   (*e*) $7\,464\,000 \times 0.000\,430\,5$
   (*f*) 6745×4977
   (*g*) $847\,200 \times 0.6383$
   (*h*) $0.000\,490\,9 \div 1\,683\,000$

3. Evaluate using powers of 10.
   (*a*) $788.9 \times 0.004\,71$
   (*b*) $0.004\,111 \times 0.009\,311$
   (*c*) $16\,440 \div 4831$
   (*d*) $0.009\,29 \div 16.52$
   (*e*) $6668 \div 496.6$
   (*f*) $0.944\,1 \times 4375$
   (*g*) $9\,441\,000 \times 0.004\,365$
   (*h*) $0.008\,414 \div 169\,000$
   (*i*) $1\,050\,000 \div 102.1 \times 0.008\,913$
   (*j*) $9840 \div 487\,500 \times 7161$

4. Now go back to page 1 and time yourself on that question.

3.(d) 5.623×10^{-4}
(g) 41 210
(h) 4.978×10^{-8}

Patterns with Powers

1. Find the pattern and complete.

$3^2 = 9 = 4 + 5$	$3^2 + 4^2 = 5^2$
$5^2 = 25 = 12 + 13$	$5^2 + 12^2 = \square^2$
$7^2 = 49 = 24 + \square$	$7^2 + \square^2 = 25^2$
$9^2 = 81 = 40 + \square$	$9^2 + \square^2 = \square^2$
$11^2 = \square = 60 + \square$	$11^2 + \square^2 = \square^2$
$13^2 = \square = \square + \square$	$\square^2 + \square^2 = \square^2$
$15^2 = \square = \square + \square$	$\square^2 + \square^2 = \square^2$

2. The value of 12^2 is 144. If we reverse the digits of 12 we get $21^2 = 441$. Notice that the digits of the square are also reversed. Find two other two-digit numbers of this type.

3. Find three different three-digit square numbers using the digits 1, 6, 9. For example, 916 is a three-digit number but is not a square.

4. Copy and complete the following patterns.

(*a*)	(*b*)
$1^2 + 1 = 2^2 - \square$	$2^2 - 1^2 = 2 + \square$
$2^2 + 2 = 3^2 - \square$	$3^2 - 2^2 = \square + 2$
$3^2 + 3 = \square^2 - \square$	$4^2 - 3^2 = 4 + \square$
$4^2 + 4 = \square^2 - \square$	$5^2 - \square^2 = \square + 4$
$5^2 + 5 = \square^2 - \square$	$6^2 - \square^2 = \square + \square$

5. Throughout history, many mathematicians have proposed theories about numbers. By substituting several different values for n, test each of the following theories to see which are true.

   Theory 1: If n is not divisible by 7, either $n^3 - 1$ or $n^3 + 1$ is divisible by 7.

   Theory 2: The expression $3n + 2$ is never equal to a square number.

   Theory 3: The number $2^n - 1$ is always a prime number.

   Theory 4: The number $n^2 - n + 41$ is always a prime number.

6. Complete this pattern.

$1^3 = 1 = 1$	$5^3 = \square = \square$
$2^3 = 8 = 3 + 5$	$6^3 = \square = \square$
$3^3 = 27 = 7 + 9 + 11$	$7^3 = \square = \square$
$4^3 = 64 = 13 + 15 + 17 + 19$	$8^3 = \square = \square$
	$9^3 = \square = \square$

Unit 2

Operations with Algebraic Terms

Algebra is generalized arithmetic, in which variables (letters) represent numbers. Like numbers, algebraic expressions can be added, subtracted, multiplied, and divided.

2.1 Get-Set Exercise

A *Addition and Subtraction*

You can simplify $7c + 3c + 5c$ by writing the terms vertically *or* horizontally.

$$\begin{array}{r} 7c \\ 3c \\ 5c \\ \hline 15c \end{array} \qquad 7c + 3c + 5c = 15c$$

ADD THE COEFFICIENTS

You must remember to consider the signs.

Simplify $-5a + 6a + 7a - 3a$

$$\begin{array}{r} -5a \\ +6a \\ +7a \\ -3a \\ \hline 5a \end{array} \qquad \text{or } -5a + 6a + 7a - 3a = 5a$$

ADD THE COEFFICIENTS AND CONSIDER THE SIGNS

Terms with the variable portions identical.

In examples where different variables or different powers of the same variable are present, you must remember to collect only *like* terms.

$$\begin{array}{l} 4m + 2n + 3m - 2n - 3n^2 \\ \quad - 2m + 5n^2 + 6n \end{array}$$

$$= \begin{array}{r} 4m + 2n \\ 3m - 2n - 3n^2 \\ -2m + 6n + 5n^2 \\ \hline 5m + 6n + 2n^2 \end{array}$$

$$\begin{array}{l} 4m + 2n + 3m - 2n - 3n^2 \\ \quad - 2m \ + 5n^2 + 6n \end{array}$$

$$= 5m + 6n + 2n^2$$

COLLECT THE n^2's

COLLECT THE n's

COLLECT THE m's

Monomial? What's that? One term, friend.

B *Multiplication: Monomials*

$5x \times 2y$ means $5 \times x \times 2 \times y = 5 \times 2 \times x \times y$
$= 10xy$

$6m^3n^2 \times (-3mn^3)$ means $6 \times m^3 \times n^2 \times (-3) \times m \times n^3$
$= 6 \times (-3) \times m^3 \times m \times n^2 \times n^3$
$= -18m^4n^5$

Then a binomial must mean two terms! You're great!

C *Multiplication: A Monomial and a Binomial*

The distributive property is used to form this product.

$$\begin{aligned} 7(5a - 3c) &= 7(5a) + 7(-3c) \\ &= 35a - 21c \\ 3a\,(5a - 3c) &= 3a(5a) + 3a(-3c) \\ &= 15a^2 - 9ac \end{aligned}$$

Exercise 2.1

Written Solutions

1. Add or subtract as indicated.

(a) $2x + 7x + 4x$
(b) $2m + 4m + 3m$
(c) $3a + (-2a) + 5a$
(d) $3r + 2r + 9r$
(e) $2c + 3d + 4c + d$
(f) $3x + 5y + 3y - 2x$
(g) $5e - 3f - 4e + f$
(h) $-5a - 3b + 2a + 6a$
(i) $2m - 5m - 6n + 3n$
(j) $7x - 4y - 3x + 9y$
(k) $5x^2 - 3x - 2x^2 + x$
(l) $7m^2 - 3m + 4m - 5m^2$
(m) $15a^2 - 7 + 4a - 9a^2 + a$
(n) $11r^2 + 9r - 4r^2 - 11r$

Remember to collect only like terms.

1(a) $13x$
(c) $6a$
(e) $6c + 4d$
(g) $e - 2f$
(h) $3a - 3b$
(j) $4x + 5y$
(l) $2m^2 + m$
(n) $7r^2 - 2r$

2. Multiply the monomials.

(a) $3a \times 2b$
(b) $2m \times 3n$
(c) $5x \times 2y$
(d) $5r \times (-2s)$
(e) $-3c(2d)$
(f) $2m^2 \times (7n^2)$
(g) $11mn(2m^2n)$
(h) $-4a^2(-3ab^3)$
(i) $25c^{-2}(-2c^2d)$
(j) $6ab(4a^2b^3)$
(k) $-5a^{-2}b^3(-5a^4b^7)$
(l) $16x^2y \times 2xy^3$
(m) $4r^2s^3(-3rs)$
(n) $-7m^7n^{-3}(3m^{-4}n^3)$
(o) $14t^3 \times 3t^{-3}$
(p) $2x \times 3x^3(-4x)$

3. Multiply.

(a) $5(2x + y)$
(b) $7(3a - 4b)$
(c) $2x(3x - 2y)$
(d) $4m(2m - 3)$
(e) $a(3b + 2c)$
(f) $-3r(2s - 5)$
(g) $3x(5x^2 - 2x + 7)$
(h) $2c(-7c^2 + 2c - 4)$

4. Multiply and simplify.

(a) $x^2 + 4x - 2(x + 3) + x(2x - 4)$
(b) $3m^2 - 2m + m(2 + m)$
(c) $3t^2 - 3t + 2t(t - 3) - 4t(3 - t)$
(d) $m^2 - 3m + 6 + m^2(3 + 2m) - m(3m - 4)$
(e) $a^2b - 4ab + 2a(ab - b) + 3b(a^2 - a)$
(f) $c^3d^2 - c^2d^3 - 3d(c^3d + c^2d^2) - 5c^2d(cd + d^2)$

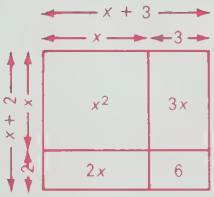
2(c) $10xy$
(f) $14m^2n^2$
(k) $25a^2b^{10}$
(n) $-21m^3$

3(b) $21a - 28b$
(f) $-6rs + 15r$

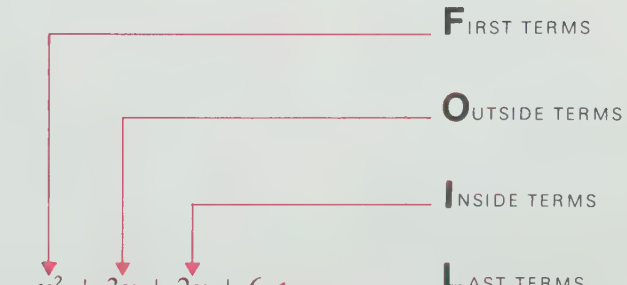
4(a) $3x^2 - 2x - 6$
(e) $6a^2b - 9ab$

2.2 Multiplying Binomials

Find the product of $(x + 2)(x + 3)$.



Total area $= x^2 + 3x + 2x + 6$
$= x^2 + 5x + 6$



$(x + 2)(x + 3) = x^2 + 3x + 2x + 6$
$= x^2 + 5x + 6$

But total area is also length $\times$ width.

$(x + 3)(x + 2)$
$\therefore (x + 3)(x + 2) = x^2 + 5x + 6$

Use FOIL!

Exercise 2.2

Written Solutions

1. Expand and simplify.
   (a) $(x+3)(x+2)$
   (b) $(c-4)(c+2)$
   (c) $(d-3)(d+5)$
   (d) $(y-4)(y-2)$
   (e) $(4x-1)(x+4)$
   (f) $(2f+2)(3f-5)$
   (g) $(2y-5)(y+7)$
   (h) $(7a+2)(2a-5)$
   (i) $(6m-5)(m+2)$
   (j) $(n+3)(3n-5)$
   (k) $(2x+3y)(x-2y)$
   (l) $(5m-2n)(3m+n)$
   (m) $(2a-3)(a-1)$
   (n) $(2t-3)(t+5)$
   (o) $(5m-6)(3m+2)$
   (p) $(3c+d)(2c-7d)$
   (q) $(3b+c)(b-2c)$
   (r) $(x-7y)(3x+5y)$

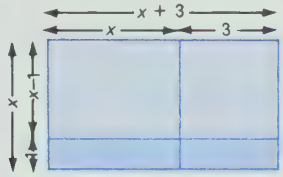
2. Determine the following products using areas and check using FOIL.
   (a) $(x+6)(x+3)$
   (b) $(2x+1)(x+2)$

3. Multiply and simplify.
   (a) $(x+2)(x-5)+(2x+1)(x-1)$
   (b) $(a+3)(a-2)+(a+1)(a-6)$
   (c) $(c-3)(2c+5)+(c-4)(3c+1)$
   (d) $(2m+1)(2m-1)-(m+2)(m-2)$
   (e) $(t-4)^2+(2t+1)(t-3)$
   (f) $(2x-3)(x+4)-(3x+1)(x-2)$
   (g) $(3d+2)(2d-3)-(2d+1)(d-1)$
   (h) $(x+3)^2-(2x-3)(2x+3)+(x+5)^2$
   (i) $(t+1)^2+(2t-3)(2t+3)-(t+3)^2$
   (j) $(3c-4d)(3c+4d)-(c-3d)^2$
   (k) $(2y+5)(3y-1)+(2y+3)(y-2)$
   (l) $(5m+2)(3m-4)-(m+2)(m-1)$
   (m) $(7c-2)^2-(2c+1)(3c-5)$
   (n) $(e+4)^2+(2e-5)(3e+1)$
   (o) $(n+6)(n-3)-(n-2)^2+(3n+1)$
   (p) $(2r+3)(3r+2)-(2r-3)^2$
   (q) $(5x-2)(5x+2)+(5x+2)^2$
   (r) $(3y-2x)(3y+2x)-(3y-2x)^2$
   (s) $(a-3c)(a+2c)-(2a+c)^2$
   (t) $(3z-4)(3z+4)+(2z+1)^2-(z-3)^2$

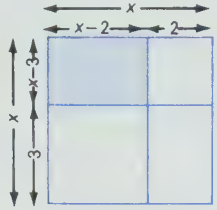
1(a) x^2+5x+6
(e) $4x^2+15x-4$
(i) $6m^2+7m-10$
(m) $2a^2-5a+3$
(q) $3b^2-5bc-2c^2$

3(a) $3x^2-4x-11$
(f) $-x^2+10x-10$
(k) $8y^2+12y-11$
(p) $2r^2+25r-3$
(t) $12z^2+10z-24$

4. Determine the product $(x + 3)(x - 1)$ using areas.



5. Determine the product $(x - 2)(x - 3)$ using areas.



2.3 Division

A *Division by a Monomial*

$$\frac{7x}{7} = x$$

$$\frac{7x + 7y}{7} = x + y$$

$$\frac{-15x^2 - 15x}{5x} = -3x - 3$$

The denominator must divide into *each term* in the numerator.

$\frac{7x+7y}{7} = \frac{\cancel{7}^{1}(x+y)}{\cancel{7}_{1}} = x+y$

OR

$\frac{7x+7y}{7} = \frac{1}{7}(7x+7y) = x+y$

B *Division by a Binomial*

Study these examples.

1. Divide $x^2 + 3x - 10$ by $x - 2$.

STEP 1

$$\begin{array}{r} x \\ x-2\,\overline{)\,x^2+3x-10} \\ \underline{x^2-2x} \\ 5x-10 \end{array}$$

How do you get this?

Remember the rule for subtraction?

STEP 2

$$\begin{array}{r} x+5 \\ x-2\,\overline{)\,x^2+3x-10} \\ \underline{x^2-2x} \\ 5x-10 \\ \underline{5x-10} \\ 0 \end{array}$$

How do you get this?

Careful here!

2. Divide $x^3 - 2x^2 - 5x + 6$ by $x - 3$.

$$
\begin{array}{r|l}
 & x^2 + x - 2 \\
\hline
x - 3 & x^3 - 2x^2 - 5x + 6 \\
 & \underline{x^3 - 3x^2} \\
 & \quad x^2 - 5x \\
 & \quad \underline{x^2 - 3x} \\
 & \quad\quad - 2x + 6 \\
 & \quad\quad \underline{- 2x + 6} \\
 & \quad\quad\quad 0
\end{array}
$$

Exercise 2.3

Written Solutions

The denominator must divide into each term of the numerator.

1. Divide.

(a) $\dfrac{4x}{4}$ (e) $\dfrac{7a - 14}{-7}$ (i) $\dfrac{5x^2 + 5x}{5x}$

(b) $\dfrac{7bc}{7c}$ (f) $\dfrac{2mn - 4}{2}$ (j) $\dfrac{2a^2b - 4ab^2}{2ab}$

(c) $\dfrac{-14xy}{-2y}$ (g) $\dfrac{15ab - 5b}{-5b}$ (k) $\dfrac{-3m^3 - 6m^2n}{-3m^2}$

(d) $\dfrac{5b + 5c}{5}$ (h) $\dfrac{27a^3 - 9a}{3a}$ (l) $\dfrac{5x^2y - 15xy^2}{5xy}$

2. Divide.

(a) $\dfrac{x^2 + 7x + 12}{x + 3}$ (e) $\dfrac{x^2 + 5x - 14}{x + 7}$

(b) $\dfrac{x^2 + 5x + 6}{x + 2}$ (f) $\dfrac{6m^2 - 13m - 5}{2m - 5}$

(c) $\dfrac{x^2 - x - 6}{x - 3}$ (g) $\dfrac{x^3 - 7x^2 + 11x - 2}{x - 2}$

(d) $\dfrac{x^2 + 3x - 10}{x + 5}$ (h) $\dfrac{3x^3 - 14x^2 + 16x - 3}{x - 3}$

3. Is $x^3 - 5x^2 + x - 1$ divisible by $x - 1$?

4. Find the remainder when $x^3 - 2x^2 - 5x + 6$ is divided by $x - 2$.

5. One factor of $a^3 + a^2 - 3a - 2$ is $a + 2$. Find the other factor.

1(d) $b + c$
(f) $mn - 2$
(i) $x + 1$
(l) $x - 3y$

2(d) $x - 2$
(h) $3x^2 - 5x + 1$

6. The area of a rectangle is $2m^3 - 3m^2 - 2m + 3$ and the width is $m - 1$. Find the length.

7. When $x - 4$ is divided into a polynomial, the quotient is $2x^2 - 3x + 3$ and the remainder is 6. Find the polynomial.

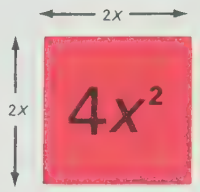
A polynomial is an expression with any number of terms.

2.4 Squaring

A *Squaring Monomials*

$(2x)^2$ means $2x \times 2x = 4x^2$
$(3x^2y)^2$ means $3x^2y \times 3x^2y = 9x^4y^2$

You simply square each factor of the monomial.



$(2x)^2$ is read "2x quantity squared".

B *Squaring Binomials*

FOIL can be used to square a binomial also, but perhaps you can find a short cut in the following examples.

$$(x + 3)^2 = (x + 3)(x + 3)$$
$$= x^2 + 3x + 3x + 9$$
$$= x^2 + 6x + 9$$

How can you get these from x and 3? Check the next one.

$$(y - 5)^2 = (y - 5)(y - 5)$$
$$= y^2 - 5y - 5y + 25$$
$$= y^2 - 10y + 25$$

How can you get -10 y from y and -5?

Try your short cut on $(5x - 2y)^2 = 25x^2 - 20xy + 4y^2$.
To square a binomial you simply square each term and then *double the product of the terms* to obtain the middle term.

The terms are 2x and -5.

$$(2x - 5)^2 = 4x^2 - 20x + 25$$

SQUARE OF TERMS

DOUBLE THE PRODUCT

Exercise 2.4

Discussion

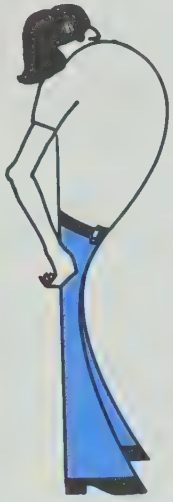
1. Square each given term.

(a) 3	(e) $-2x$	(i) $\frac{-1}{2}$	(m) 0.2
(b) $-b$	(f) x^2	(j) $\frac{1}{3}$	(n) 0.8
(c) x	(g) y	(k) $-\frac{2}{3}x$	(o) $0.6x$
(d) $3x$	(h) $2x^2$	(l) $\frac{3}{5}x$	(p) $-\frac{7}{8}x$

2. Find twice the product of the given terms.

(a) x, 3	(g) 6, $-3x$	(m) $3m$, $-2r$
(b) $5x$, 3	(h) $4x$, -2	(n) $2q$, $-3p$
(c) $2r$, -3	(i) x, 7	(o) $-2r$, $-3s$
(d) $7m$, $2n$	(j) $4x$, -1	(p) a, $4b$
(e) $9x$, -3	(k) $2x$, $-3r$	(q) $-3a$, $4c$
(f) $2x$, $-4y$	(l) $2t$, $4s$	(r) $2d$, $-3e$

No, man! They mean multiply!



Written Solutions

3. Expand.

(a) $(a+b)^2$	(i) $(5a+2b)^2$	(q) $(6d+7e)^2$
(b) $(x-2)^2$	(j) $(p-2q)^2$	(r) $(3c-2d)^2$
(c) $(2y+5)^2$	(k) $(4m+1)^2$	(s) $(3a-5b)^2$
(d) $(3z-1)^2$	(l) $(3t-2r)^2$	(t) $(2e+3f)^2$
(e) $(3c-4)^2$	(m) $(5x-3)^2$	(u) $(4f+2q)^2$
(f) $(5x-2y)^2$	(n) $(2z-9)^2$	(v) $(7+4r)^2$
(g) $(3x+1)^2$	(o) $(3x-4y)^2$	(w) $(4m-9)^2$
(h) $(7m-2n)^2$	(p) $(6p-4q)^2$	(x) $(7e-2f)^2$

2.5 A Special Product

When the sum and difference of two quantities are multiplied, the result is always the difference of their squares. Consider the sum and difference of two terms, x and 2.

Product of sum and difference of same two quantities.

$$(x+2)(x-2) = x^2 - 2x + 2x - 4$$
$$= x^2 - 4$$

Similarly, $(3a+2b)(3a-2b) = 9a^2 - 6ab + 6ab - 4b^2$
$$= 9a^2 - 4b^2$$

Difference of two squares.

Why do the middle terms drop out?

Exercise 2.5

Discussion

1. State the difference of the squares of the given quantities.

(*a*) x, 2	(*f*) $4x$, 3	(*k*) $7b$, 2
(*b*) m, 3	(*g*) $2m$, 5	(*l*) $2x$, $3y$
(*c*) $2a$, 7	(*h*) $7p$, 3	(*m*) $4m$, $3n$
(*d*) $4e$, 3	(*i*) $5a$, 2	(*n*) 4, $3r$
(*e*) $2r$, s	(*j*) $3e$, 7	(*o*) 3, $2x$

2. Each part is a difference of squares of two quantities. What are the quantities?

(*a*) $x^2 - 9$	(*d*) $9t^2 - 1$	(*g*) $25b^2 - 4$
(*b*) $m^2 - 4$	(*e*) $16a^2 - 9$	(*h*) $4n^2 - 49$
(*c*) $4r^2 - 9$	(*f*) $4c^2 - 25$	(*i*) $x^2 - 25y^2$

3. What should you multiply these by to give a difference of two squares? State the product in each case.

(*a*) $(x + 3)$	(*d*) $(3e + 5)$
(*b*) $(y - 2)$	(*e*) $(2f - 7)$
(*c*) $(2m + 5)$	(*f*) $(g - 2)$

Written Solutions

4. State the product.

(*a*) $(x - 4)(x + 4)$	(*g*) $(3x - 8y)(3x + 8y)$
(*b*) $(y - 7)(y + 7)$	(*h*) $(6q - 5r)(6q + 5r)$
(*c*) $(2x - 5y)(2x + 5y)$	(*i*) $(4 - q)(4 + q)$
(*d*) $(3m + 5)(3m - 5)$	(*j*) $(5 - 2q)(5 + 2q)$
(*e*) $(z - 2q)(z + 2q)$	(*k*) $(6 + t)(6 - t)$
(*f*) $(2t - 7)(2t + 7)$	(*l*) $(7c + 2e)(7c - 2e)$

5. Expand.

(*a*) $(2x + 3)(2x - 3)$	(*h*) $(4x + 5)(4x - 5)$
(*b*) $(3x - 2)(3x + 2)$	(*i*) $(7x - 2y)(7x + 2y)$
(*c*) $(5x - 7y)(5x + 7y)$	(*j*) $(3x - 1)(3x + 1)$
(*d*) $(x - 10y)(x + 10y)$	(*k*) $(7x + y)(7x - y)$
(*e*) $(2a + 5)(2a - 5)$	(*l*) $(6x - 5y)(6x + 5y)$
(*f*) $(3m - 2n)(3m + 2n)$	(*m*) $(2m - n)(2m + n)$
(*g*) $(2c - 3y)(2c + 3y)$	(*n*) $(3x + 2c)(3x - 2c)$

4(b) $y^2 - 49$
(e) $z^2 - 4q^2$
(h) $36q^2 - 25r^2$
(k) $36 - t^2$

5(a) $4x^2 - 9$
(d) $x^2 - 100y^2$
(g) $4c^2 - 9y^2$
(h) $16x^2 - 25$
(k) $49x^2 - y^2$

2.6 Cube of a Binomial

$$(a+b)^3 = (a+b)(a+b)^2$$
$$= (a+b)(a^2+2ab+b^2)$$
$$= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3$$
$$= a^3+3a^2b+3ab^2+b^3$$
$$\therefore (a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

Squaring.
Multiplying.
Collect like terms.
The cube of each term
plus
3 times the square of one term times the other term.

Use the pattern from above.

$$(a-b)^3 = a^3+3(a^2)(-b)+3(a)(-b)^2+(-b)^3$$
$$= a^3-3a^2b+3ab^2-b^3$$

Example 1

Cube of each term plus 3 times the square of one term times the other term.

$$(2x+3y)^3 = (2x)^3+3(2x)^2(3y)+3(2x)(3y)^2+(3y)^3$$
$$= 8x^3+3(4x^2)(3y)+3(2x)(9y^2)+(27y^3)$$
$$= 8x^3+36x^2y+54xy^2+27y^3$$

Example 2

$$(m-2n)^3 = m^3+3(m)^2(-2n)+3(m)(-2n)^2+(-2n)^3$$
$$= m^3-6m^2n+12mn^2-8n^3$$

Exercise 2.6

Written Solutions

1. Determine the cube of both terms.

(a) x, y	(e) $r, 2s$	(i) $2v, w$
(b) $2m, n$	(f) $c, -3$	(j) $3x, -2$
(c) $a, -2$	(g) $2r, -s$	(k) $d, -2$
(d) $3x, 2y$	(h) $2a, -b$	(l) $5a, 4c$

2. Calculate three times the square of the first term times the second.

(a) x, y	(e) $2e, -3f$	(i) $2x, y$
(b) $2a, b$	(f) $y, -2z$	(j) $5m, n$
(c) $m, -2$	(g) $3c, -d$	(k) $2d, -e$
(d) $2t, -u$	(h) $m, -3$	(l) $r, -3s$

2 (e)
$3(2e)^2(-3f)$
$= 3(4e^2)(-3f)$
$= -36e^2f$

2(a) $3x^2y$
(d) $-12t^2u$
(i) $12x^2y$
(l) $-9r^2s$

3. Calculate three times the first term times the square of the second.

(*a*) x, y	(*e*) $a, 2b$	(*i*) $2e, -f$
(*b*) $x, -y$	(*f*) $e, -2f$	(*j*) $a, -2$
(*c*) $m, -2$	(*g*) $2p, 3$	(*k*) $3r, -s$
(*d*) $a, -3b$	(*h*) $2t, -3$	(*l*) $3m, -n$

4. Determine the cube of the following binomials.

(*a*) $x + y$	(*j*) $3y + z$	(*s*) $f + 2g$
(*b*) $a - b$	(*k*) $2p - 3q$	(*t*) $3p - 5$
(*c*) $2a + b$	(*l*) $m - 3n$	(*u*) $5x - y$
(*d*) $a - 2$	(*m*) $c - 2d$	(*v*) $x + 5y$
(*e*) $x - 3y$	(*n*) $2c + 3$	(*w*) $2q + 3f$
(*f*) $2x + y$	(*o*) $2m + n$	(*x*) $4f - e$
(*g*) $3a + b$	(*p*) $2f - g$	(*y*) $e + 3$
(*h*) $2r - s$	(*q*) $e - 2f$	(*z*) $5x - 2y$
(*i*) $a - 3$	(*r*) $2m - n$	

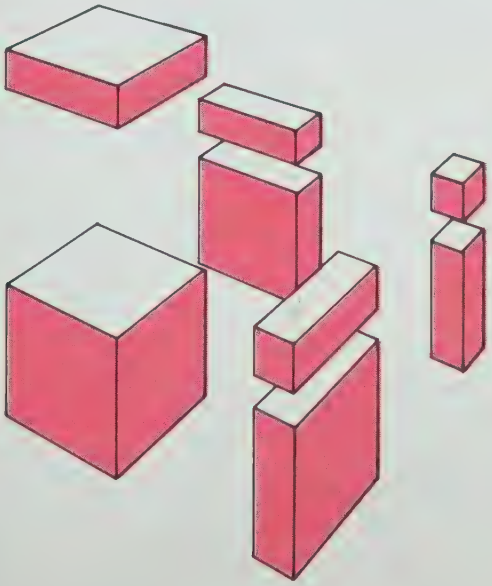
3(a) $3xy^2$
(e) $12ab^2$
(i) $6ef^2$

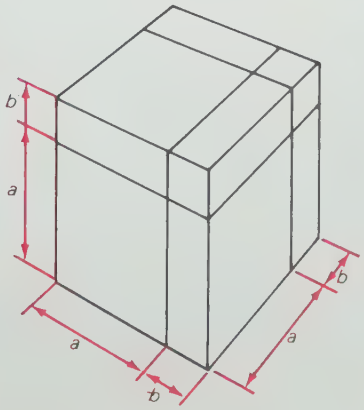
4(b) $a^3 - 3a^2b + 3ab^2 - b^3$
(g) $27a^3 + 27a^2b + 9ab^2 + b^3$
(l) $m^3 - 9m^2n + 27mn^2 - 27n^3$
(q) $e^3 - 6e^2f + 12ef^2 - 8f^3$
(y) $e^3 + 9e^2 + 27e + 27$

5. Using light cardboard, construct the following solids.

12 cm × 12 cm × 12 cm	1 only
4 cm × 4 cm × 4 cm	1 only
12 cm × 12 cm × 4 cm	3 of these
12 cm × 4 cm × 4 cm	3 of these

Put these solids together and you have a model of $(a + b)^3$, the cube of a binomial.





The Binomial Expansion

We have seen that $(a + b)^2 = a^2 + 2ab + b^2$
and $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

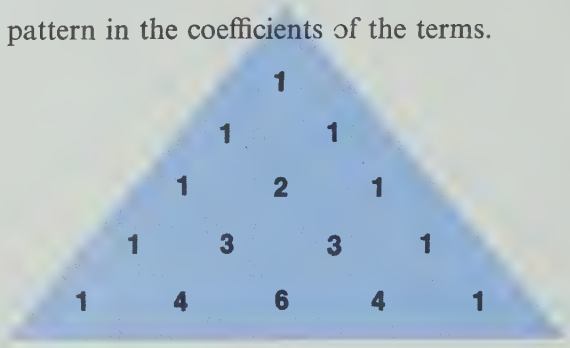
Now let us examine $(a + b)^4$.

$$\begin{aligned}(a + b)^4 &= (a + b)^2(a + b)^2 \\ &= (a^2 + 2ab + b^2)(a^2 + 2ab + b^2) \\ &= a^4 + 2a^3b + a^2b^2 + 2a^3b + 4a^2b^2 + 2ab^3 + a^2b^2 \\ &\qquad + 2ab^3 + b^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

Now let's look for a pattern.

$$\begin{aligned}(a + b)^0 &= 1 && \leftarrow \text{1 TERM} \\ (a + b)^1 &= a + b && \leftarrow \text{2 TERMS} \\ (a + b)^2 &= a^2 + 2ab + b^2 && \leftarrow \text{3 TERMS} \\ (a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 && \leftarrow \text{4 TERMS} \\ (a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 && \leftarrow \text{5 TERMS}\end{aligned}$$

There is a pattern in the coefficients of the terms.



This is called Pascal's triangle. Note that each number is the sum of the two numbers immediately above it. Could you predict the coefficients of the terms for $(a + b)^5$?

The variables in each term also follow a definite pattern. In $(a + b)^4$, the terms are a^4, a^3b, a^2b^2, ab^3, b^4. Each term contains 4 factors.

a^4 – four a's
a^3b – three a's, one b
a^2b^2 – two a's, two b's
ab^3 – one a, three b's
b^4 – four b's

Could you predict the terms for $(a + b)^5$?

Summary

In $(a + b)^4$

- There are 5 terms.
- The coefficients in order are 1, 4, 6, 4, 1.
- Each term contains 4 factors.

Example

Expand $(2x - y)^6$ using the patterns suggested.

Solution

—There will be 7 terms in the expansion.
—The coefficients in order will be 1, 6, 15, 20, 15, 6, 1.
—Each term will contain 6 factors.

$$\therefore (2x - y)^6 = (2x)^6 + 6(2x)^5(-y) + 15(2x)^4(-y)^2 + 20(2x)^3(-y)^3 + 15(2x)^2(-y)^4 + 6(2x)(-y)^5 + (-y)^6$$
$$= 64x^6 - 192x^5y + 240\,x^4y^2 - 160x^3y^3 + 60x^2y^4 - 12xy^5 + y^6$$

Try these.

1. Continue Pascal's triangle for 6 more rows of numbers.
2. Expand the following using the patterns suggested above.
   (*a*) $(a + b)^5$ (*c*) $(2a + b)^6$ (*e*) $(x - 2y)^6$
   (*b*) $(a + b)^6$ (*d*) $(2x - y)^5$ (*f*) $(a - 1)^7$
3. Find the following.
   (*a*) The fifth term in $(a + 2b)^8$.
   (*b*) The third term in $(2x - 5)^5$.
   (*c*) The fourth term in $(3m - 1)^7$.
   (*d*) The last term in $(5x - y)^8$.
   (*e*) The first term in $(m - 7n)^{15}$.

Unit 3

Factoring Algebraic Expressions

In Unit 2 we were concerned with multiplication of polynomials (*expanding*). We now consider the inverse operation, *factoring*.

Expanding	*Factoring*
$5(x + y) = 5x + 5y$	$5x + 5y = 5(x + y)$

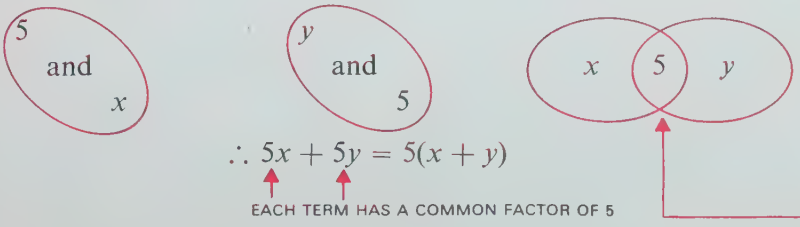
To factor means to express as a product.

3.1 Common Factoring

Study these examples.

1. Factor $5x + 5y$.

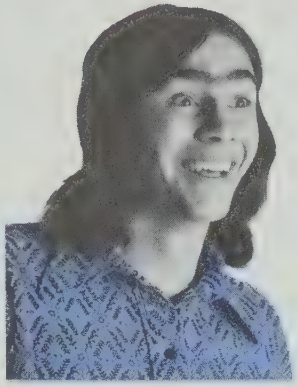
Factors of $5x$. Factors of $5y$. Factors of $5x$ and $5y$.



$\therefore\ 5x + 5y = 5(x + y)$

EACH TERM HAS A COMMON FACTOR OF 5

5 is the common factor.



Aha! Distributive Property again!

2. Find the common factor in $14x - 7y$.

$$14x - 7y = 7(2x - y)$$

DIVIDE

The common factor is a factor of *each* term.

$$3c - 6d + 9e = 3(c - 2d + 3e),$$

and $5x^3 - 10x^2 - 25x = 5x(x^2 - 2x - 5)$.

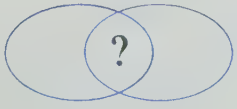
But $2x^2 + 4x - 3$ cannot be factored.

Exercise 3.1

1. State the highest common factor. Find the other factor.

	(*a*)	$3x + 3y$	(*l*) $2m^3n + 4mn^3 + 6mn$
	(*b*)	$7m - 7n$	(*m*) $3a + 6b + 3c$
	(*c*)	$5a - 5b$	(*n*) $5x - 15x^2$
	(*d*)	$2c - 2d + 2e$	(*o*) $2y - 4z$
	(*e*)	$10a - 5b$	(*p*) $3e^2 - 15e$
	(*f*)	$7p - 14q$	(*q*) $7x^2y + 14xy^2$
*	(*g*)	$3a^2 + 3ab$	(*r*) $12ab + 3bc$
	(*h*)	$5x^2 + 5xy$	(*s*) $27m^2n - 9m$
	(*i*)	$8m^3 + 16m^2n$	(*t*) $16a^2b + 32b^2c$
	(*j*)	$11e^3 + 22e^2f$	(*u*) $35abc + 70bc + 105ac$
	(*k*)	$6x^2y + 9x^3y^2$	(*v*) $14m^3n^2 - 28mn^2$

* Watch out for (g). Use the highest common factor.



2. Suppose we look at $x(x + 3) - 2(x + 3)$ as a binomial.
   (*a*) What are the terms?
   (*b*) What is the common factor of each of the terms?
   (*c*) If you divide this common factor into each of the two terms of the binomial, what is the resulting second factor?

3. Factor.

	(*a*)	$5(a + b) + 3c(a+b)$	(*e*) $2p(m - 2n) - 3q(m - 2n)$
	(*b*)	$x(a + b) + y(a + b)$	(*f*) $2a(3c - 1) - b(3c - 1)$
	(*c*)	$2x(a - b) - 3y(a - b)$	(*g*) $4r(3a + 2b) - 3s(3a + 2b)$
	(*d*)	$3x(2r - 1) + 2y(2r - 1)$	(*h*) $7x(2z - m) + (2z - m)$

1(a) 3, $(x + y)$
(f) 7, $(p - 2q)$
(j) $11e^2$, $(e + 2f)$
(n) $5x$, $(1 - 3x)$
(r) $3b$, $(4a + c)$

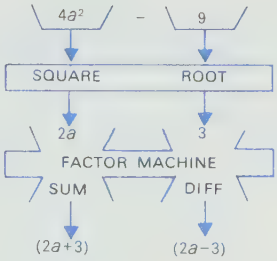
3(c) $(a - b)(2x - 3y)$
(g) $(3a + 2b)(4r - 3s)$

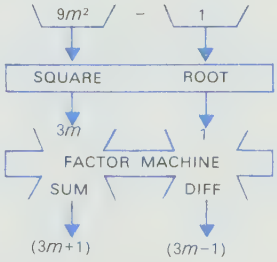
3.2 Factoring a Difference of Squares

$x^2 - y^2$ is a difference of the perfect squares x^2 and y^2. Binomials of this form always result from a special product, a topic that was studied in Unit 2. (See section 2.5)

In order to factor a difference of squares:

1. Determine the square root of each of the perfect squares.
2. Form two binomials using each square root in each binomial.
3. Make sure that the signs between the two terms in each binomial factor are different.





Example 1

$9m^2 - 1 = (3m + 1)(3m - 1)$

SQUARE ROOT, SQUARE ROOT, SUM, DIFFERENCE

$3m$ 1

Example 2

$(x + 2)^2 - 9 = (x + 2 + 3)(x + 2 - 3) = (x + 5)(x - 1)$

SQUARE ROOT, SQUARE ROOT, SUM, DIFFERENCE

$x + 2$ 3

Example 3

$25 - (x + 3)^2 = (5 + [x + 3]^*)(5 - [x + 3]^*) = (8 + x)(5 - x - 3)$

$= (8 + x)(2 - x)$

SQUARE ROOT, SUM, DIFFERENCE

5 $x + 3$

* Always insert brackets when a binomial is the second term.

Exercise 3.2

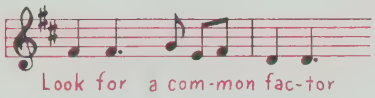
Discussion

1. Each binomial given is a difference of the squares of two quantities.
   - State the square root of each quantity.
   - State the sum of these square roots.
   - State the difference of these square roots.
   - State the product of this sum and difference.

(a) $a^2 - 9$ (d) $4x^2 - 1$ (g) $9t^2 - 25$

(b) $m^2 - 25$ (e) $25b^2 - 1$ (h) $4c^4 - 9$

(c) $c^2 - 36$ (f) $4m^2 - n^2$ (i) $16a^2 - 25b^2$



Written Solutions

2. Factor fully.

(a) $2a^2 - 18$
(b) $3m^2 - 75$
(c) $2c^2 - 8$
(d) $5x^2 - 20$
(e) $3b^2 - 12$
(f) $5m^3 - 45mn^2$
(g) $ab^2 - a$
(h) $4x^2 - 16$
(i) $6t^2 - 54$
(j) $49 - m^2$
(k) $36 - b^2$
(l) $72 - 2t^2$
(m) $4m^2 - 9$
(n) $4m^2 - 16$
(o) $5x^2 - 125$
(p) $rs^2 - 9r$
(q) $16t^2 - 64$
(r) $c^2 - 49$
(s) $3d^2 - 147$
(t) $7e^2 - 63$
(u) $27 - 3m^2$
(v) $12a - 27a^3$
(w) $a^2 - 1$
(x) $625x^2 - 25$

Careful in (f) and (j)

3. Factor fully.

(a) $x^2 - 16$
(b) $(x - 2)^2 - 16$
(c) $m^2 - 25$
(d) $(m + 3)^2 - 25$
(e) $25x^2 - 9$
(f) $25 - (x + 3)^2$
(g) $49 - (y + 6)^2$
(h) $(a + 2)^2 - 49$
(i) $(b + 3)^2 - 25$
(j) $(x + 3)^2 - (y - 2)^2$
(k) $(t - 3)^2 - (r + 2)^2$
(l) $(c + 2)^2 - 4$
(m) $(m + 3)^2 - (n - 2)^2$
(n) $36 - (f + 4)^2$

2(a) $2(a + 3)(a - 3)$
(g) $a(b + 1)(b - 1)$
(m) $(2m + 3)(2m - 3)$
(s) $3(d + 7)(d - 7)$
3(b) $(x + 2)(x - 6)$
(h) $(a + 9)(a - 5)$
(n) $(10 + f)(2 - f)$

No, a perfect square is not anyone over 30.

3.3 Factoring a Perfect Square Trinomial

Trinomials often result from the product of two binomials.

$$(x + 3)(x + 5) = x^2 + 8x + 15$$
$$(x - 3)^2 = x^2 - 6x + 9$$
$$(2x + 3)(x - 2) = 2x^2 - x - 6$$
$$(x + 5)(x + 5) = x^2 + 10x + 25$$

The easiest trinomial to factor is the perfect square trinomial that is formed by squaring a binomial.

1. Factor $x^2 + 10x + 25$.
   (*a*) Notice that the first and last terms are prefect squares. Their square roots are x and 5.
   (*b*) The product of x and 5, doubled, is the middle term $10x$.
   $\therefore x^2 + 10x + 25 = (x + 5)^2$

2. Factor $y^2 - 4y + 4$.
   (*a*) y^2 and 4 have square roots, y and 2.
   (*b*) $(y \times 2)2 = 4y$
   $\therefore y^2 - 4y + 4 = (y - 2)^2$

The sign for the factor comes from the middle term.

Exercise 3.3

Discussion

1. Determine the missing centre term for the following perfect square trinomials.
   (*a*) $x^2 + \ldots + 4$
   (*b*) $y^2 + \ldots + 9$
   (*c*) $a^2 + \ldots + 36$
   (*d*) $x^2 - \ldots + 1$
   (*e*) $m^2 - \ldots + 49n^2$
   (*f*) $4m^2 + \ldots + 25$
   (*g*) $25t^2 + \ldots + 9r^2$
   (*h*) $9a^2 - \ldots + 16$

A perfect square trinomial is the square of a binomial.

2. Determine the missing last term for the following perfect square trinomials.
   (*a*) $x^2 + 4x + \ldots$
   (*b*) $y^2 - 10y + \ldots$
   (*c*) $m^2 - 14m + \ldots$
   (*d*) $a^2 + 8a + \ldots$
   (*e*) $b^2 - 6a + \ldots$
   (*f*) $t^2 - 12t + \ldots$
   (*g*) $4b^2 + 16b + \ldots$
   (*h*) $9f^2 - 6f + \ldots$

$a^2 + 2ab + b^2 = (a + b)^2$

PERFECT SQUARE TRINOMIAL — SQUARE OF A BINOMIAL

3. How do you check if a trinomial can be factored to the square of a binomial?

4. If a trinomial is a perfect square how do you decide on the sign separating the two terms?

5. Which of the following are the squares of a binomial?
   (*a*) $x^2 + 12x + 36$
   (*b*) $x^2 - 14x + 49$
   (*c*) $x^2 - 5x + 25$
   (*d*) $4x^2 - 12x + 5$
   (*e*) $25a^2 + 20a + 4$
   (*f*) $16b^2 + 24b + 9$
   (*g*) $9c^2 + 5c + 6$
   (*h*) $4d^2 - 12d + 9$
   State the factors where possible.

Written Solutions

6. Factor.

(a) $a^2 + 4a + 4$
(b) $7b^2 + 28b + 28$
(c) $c^2 - 6c + 9$
(d) $3d^2 + 18d + 27$
(e) $e^2 + 2e + 1$
(f) $11f^2 - 66f + 99$
(g) $g^2 + 10gh + 25h^2$
(h) $h^2 + 50h + 625$
(i) $2i^2 + 16it + 32t^2$
(j) $2j^2 + 48j + 288$
(k) $k^3 - 14k^2 + 49k$
(l) $3k^3 - 24k^2 + 48k$
(m) $m^2n - 4mn + 4n$
(n) $n^2 + 6np + 9p^2$
(o) $5a^2 + 20a + 20$
(p) $16p^2 - 48p + 36$
(q) $3q^2 + 36q + 108$
(r) $48r^2 + 72rt + 27t^2$
(s) $25s^2 - 20sf + 4f^2$
(t) $2t^2 - 28t + 98$
(u) $10u^3 - 20u^2 + 10u$
(v) $4v^2 - 52v + 169$
(w) $w^2 - 18w + 81$
(x) $49x^2 + 28x + 4$
(y) $4y^2 + 44y + 121$
(z) $81z^2 + 18z + 1$

6(a) $(a + 2)^2$
(f) $11(f - 3)^2$
(k) $k(k - 7)^2$
(p) $4(2p - 3)^2$
(u) $10u(u - 1)^2$
(z) $(9z + 1)^2$

3.4 Factoring by Grouping

Sometimes polynomials must be grouped in order to be factored. When grouping, you must know why you are grouping, because there is a definite purpose behind the process.

After grouping, a common binomial factor must result, or you don't use this method!

A *Grouping in Pairs (To obtain a common factor)*

Notice that $a(x + y) - b(x + y)$ is a binomial in which each term has a common binomial factor $(x + y)$. This could be common factored.

$$a(x + y) - b(x + y) = (x + y)(a - b)$$

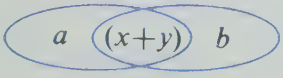
However, this example might have been met in the less obvious form of

$$ax + ay - bx - by$$

The fact that two terms have a factor "a" and two terms have a factor "$-b$" might suggest to group in pairs and common factor each of the pairs.

$$\begin{aligned} ax + ay - bx - by &= a(x + y) - b(x + y) \\ &= (x + y)(a - b) \end{aligned}$$

Since the *common binomial factor* results in each term, we know we are on the right track.



Incidentally, you might have said that in

$$ax + ay - bx - by$$

two terms have a factor of "x" and two terms have a factor of "y" and as a result, you would have grouped differently.

$$ax + ay - bx - by = x(a - b) + y(a - b)$$

But you still get a *common binomial factor*,

$$\therefore\ ax + ay - bx - by = x(a - b) + y(a - b)$$
$$= (a - b)(x + y)$$

and you get the same result!

1. Factor $mn + mp + qn + qp$.

$$mn + mp + qn + qp = m(n + p) + q(n + p)$$
$$= (n + p)(m + q)$$

Could you do it by grouping different pairs of terms?

2. Factor $x^3 - x^2 + xy - y$.

$$x^3 - x^2 + xy - y = x^2(x - 1) + y(x - 1)$$
$$= (x - 1)(x^2 + y)$$

3. Factor $mn - x - m + nx$.

$$mn - x - m + nx = mn - m + nx - x$$
$$= m(n - 1) + x(n - 1)$$
$$= (n - 1)(m + x)$$

B *Grouping 3 and 1 (To obtain a difference of squares)**

1. Factor $a^2 + 4ab + 4b^2 - 1$.

Notice that the polynomial has three terms that form a perfect square.

Thus $a^2 + 4ab + 4b^2 - 1 = (a^2 + 4ab + 4b^2) - 1$

and the three terms of the perfect square have been grouped together.

But $(a^2 + 4ab + 4b^2) - 1 = (a + 2b)^2 - 1$,

which is a difference of squares, and

$$(a + 2b)^2 - 1 = (a + 2b + 1)(a + 2b - 1)$$

2. Factor $x^2 + 6xy + 9y^2 - 4$.

$$x^2 + 6xy + 9y^2 - 4 = (x^2 + 6xy + 9y^2) - 4$$
$$= (x + 3y)^2 - 4$$
$$= (x + 3y + 2)(x + 3y - 2)$$

These seem "to go together". You know you're on the right track when the common binomial factor shows up.

* You must get a difference of squares, or you don't use this method.

Sum and difference of square roots.

Exercise 3.4

Discussion

Is there more than one way? Try it!

1. Which terms of $x^3 - 3x^2 + 3y^2 - xy^2$ seem "to go together" in pairs?
2. When you group a polynomial in pairs, what *must* happen in order for the factoring to be completed?
3. Find three terms in $t^2 + 4tr - 9 + 4r^2$ that will form a perfect square. If they are grouped together, will you be able to form a difference of squares?
4. When you group a four-term polynomial by grouping three terms together (3 and 1 grouping), what must the three terms form? What must the entire polynomial be changed into by the grouping?

Written Solutions

5. Factor by grouping in pairs to obtain a common factor.
   (*a*) $2ay + 2ax + 3by + 3bx$
   (*b*) $ax + 2x - ay - 2y$
   (*c*) $2m^2 + 2mn - m - n$
   (*d*) $2x^2 + 3x - 3y - 2xy$
   (*e*) $my - 3m + y - 3$
   (*f*) $2m^2 - 3n + 3m - 2mn$
   (*g*) $st + 3 - t - 3s$
   (*h*) $6 - 2y + xy - 3x$
   (*i*) $2x^2 - 3x - 2xy + 3y + 2xz - 3z$
   (*j*) $5am - m - 5an + n - 5a + 1$
   * (*k*) $2x^3 - 2x^2y - 3x + 3y - 3x^2 + 3xy$
   * (*l*) $ac^2 + 3b - 3 - 3a + c^2 + 3ab$

* Careful with these two.

6. Factor by grouping to obtain a difference of squares (3 and 1).
   (*a*) $c^2 + 4cd + 4d^2 - 4$
   (*b*) $a^2 + 6ab + 9b^2 - 9$
   (*c*) $f^2 - 4fg + 4g^2 - 1$
   (*d*) $4m^2 + 4mn + n^2 - 25$
   (*e*) $4c^2 - 12cd + 9d^2 - 9$
   (*f*) $25 - a^2 + 2ab - b^2$
   (*g*) $16x^2 + 8xy + y^2 - 49$
   (*h*) $4a^2 + 4ab - 4 + b^2$
   (*i*) $81x^2 + 18xy - 4 + y^2$
   (*j*) $f^2 - 4fh - 1 + 4h^2$

5(a) $(x + y)(2a + 3b)$
(d) $(2x + 3)(x - y)$
(h) $(3 - y)(2 - x)$
(l) $(a + 1)(c^2 + 3b - 3)$

6(a) $(c + 2d + 2)(c + 2d - 2)$
(d) $(2m + n + 5)(2m + n - 5)$
(g) $(4x + y + 7)(4x + y - 7)$
(j) $(f - 2h + 1)(f - 2h - 1)$

3.5 Further Factoring of Trinomials

Trinomials that are not perfect squares are factored either by inspection or by using the method of decomposition.

(*a*) Factor $x^2 + 8x + 15$.

Decomposition

① Multiply the coefficient of x^2 by the third term.
$1 \times 15 = 15$

② Decompose the coefficient of x into two numbers that multiply to $+15$ and add to $+8$ ($+5$ and $+3$)
$x^2 + 8x + 15$
$= x^2 + 3x + 5x + 15$

③ Then use 2 and 2 grouping.
$(x^2 + 3x) + 5x + 15$
$= x(x + 3) + 5(x + 3)$
$= (x + 3)(x + 5)$

Inspection

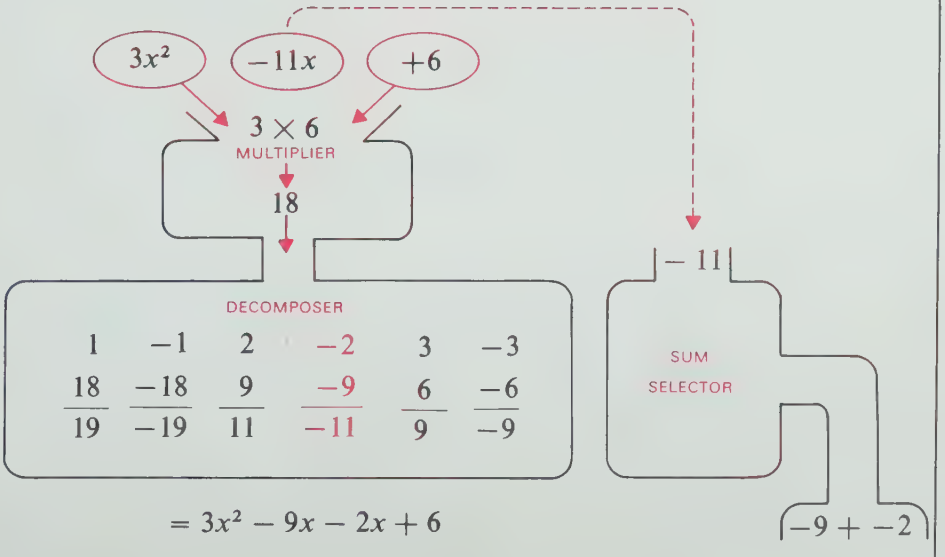
① The first term in the binomials must be x.
$(x \quad)(x \quad)$*

② The second terms must multiply to 15.*
$(x \quad 15)(x \quad 1)$ or

$(x \quad 5)(x \quad 3)$

③ Check both cases in ② for the centre term.
$\therefore (x + 5)(x + 3)$

(*b*) Factor $3x^2 - 11x + 6$.



$= 3x^2 - 9x - 2x + 6$

Cheer up. You'll soon be good at this.

* Why?

* Why?

Leave the signs for now.

(c) Factor $3x^2 - 11x + 6$.

Decomposition

$3x^2 - 11x + 6$

1. $3 \times 6 = 18$
2. Find numbers that multiply to $+18$ and add to -11. Try -9 and -2.
   $3x^2 - 11x + 6$
   $= 3x^2 - 9x - 2x + 6$
3. 2 and 2 grouping gives
   $(3x^2 - 9x) - (2x - 6)$
   $= 3x(x - 3) - 2(x - 3)$
   $= (x - 3)(3x - 2)$

Inspection

If $3x^2 - 11x + 6$ factors;

1. The first terms must be $3x$ and x.
   $(3x \quad)(x \quad)$ ← Why?
2. The second terms must multiply to 6. ← Why?
   Here are the possible factors.
   $(3x \quad 6)(x \quad 1)$
   or $(3x \quad 1)(x \quad 6)$
   or $(3x \quad 3)(x \quad 2)$
   or $(3x \quad 2)(x \quad 3)$
3. Check the centre term.
   $\therefore (3x - 2)(x - 3)$

(d) Try $6a^2 - a - 15$.

Decomposition

1. Numbers must multiply to $6 \times -15 = -90$.
2. Numbers must add to -1.
   The numbers are -10 and 9.
3. Group $6a^2 - a - 15$
   $= 6a^2 - 10a + 9a - 15$
   $= 2a(3a - 5) + 3(3a - 5)$
   $= (3a - 5)(2a + 3)$

Inspection may be very lengthy with this one.

Exercise 3.5

Discussion

1. For each trinomial given:
   - name the possible pairs of factors for the first term,
   - name the possible pairs of factors for the last term,
   - decide upon the sign in each binomial factor, and
   - check factors using the middle term.

   (a) $x^2 + 5x + 6$
   (b) $x^2 + 6x + 5$
   (c) $x^2 - 5x + 6$
   (d) $x^2 - 6x + 5$
   (e) $x^2 + x - 6$
   (f) $x^2 + 4x - 5$
   (g) $x^2 - x - 6$
   (h) $x^2 - 4x - 5$

These are easy!

Written Solutions

2. Factor.

(a) $x^2 + 7x + 10$
(b) $x^2 - 14x + 24$
(c) $a^2 - 3a - 10$
(d) $b^2 - 4b + 4$
(e) $m^2 + 2m - 8$
(f) $x^2 + 9x + 18$
(g) $p^2 - 6p + 9$
(h) $ax^2 + 4ax - 21a$
(i) $c^2 - 5c - 14$
(j) $2d^2 + 22d + 60$
(k) $s^2 + 4s - 12$
(l) $e^2 - 3e - 28$
(m) $3n^2 + 3n - 216$
(n) $x^2 + 10x + 25$
$y^2 - 14y + 49$
(p) $5r^2 + 55r + 150$
(q) $2a^2 + 24a + 72$
(r) $t^2 + 2t - 48$
(s) $m^2n - mn - 6n$
(t) $x^2y - 8xy + 16y$
(u) $a^2 + 13a + 42$
(v) $z^3 - 2z^2 - 35z$
(w) $y^2 + 6y - 7$
(x) $m^2 - 22m + 121$
(y) $x^2 + 3x - 40$
(z) $n^2 + 24n - 81$

3. For each trinomial given:
- name the possible pairs of factors for the first term,
- name the possible pairs of factors for the last term,
- decide upon the sign in each binomial factor, and
- check factors by using the middle term.

(a) $3a^2 + 5a - 2$
(b) $8b^2 + 14b + 3$
(c) $10x^2 + 23x - 21$
(d) $6r^2 - 31r + 5$

Now try each by Decomposition.

4. Factor, by Inspection.

(a) $2x^2 + 7x + 3$
(b) $2r^2 + 5r - 3$
(c) $3x^2 - 4x - 7$
(d) $3s^2 + 22s + 7$
(e) $5a^2 - 11a + 2$
(f) $5a^2 - 7a + 2$
(g) $5q^2 - 9q - 2$
(h) $7c^2 - 5c - 2$
(i) $7c^2 + 13c - 2$
(j) $5x^2 - 8x + 3$

5. Factor, by Decomposition.

(a) $3t^2 + 3t - 6$
(b) $5t^2 + 12t - 9$
(c) $3r^2 - 17r - 6$
(d) $9x^2 + 8x - 1$
(e) $12y^2 + 15y + 3$
(f) $5c^2 + c - 4$
(g) $6a^2 - 11a - 7$
(h) $3x^2 + 25x + 8$
(i) $3a^2 - 10a + 8$
(j) $5r^2 - 8r - 4$

Watch for
-common factors
-perfect squares.

2(a) $(x + 2)(x + 5)$
(f) $(x + 6)(x + 3)$
(k) $(s + 6)(s - 2)$
(p) $5(r + 5)(r + 6)$
(u) $(a + 7)(a + 6)$
(z) $(n + 27)(n - 3)$

4(a) $(2x + 1)(x + 3)$
(e) $(5a - 1)(a - 2)$
(f) $(5a - 2)(a - 1)$
(j) $(5x - 3)(x - 1)$

5(a) $3(t + 2)(t - 1)$
(e) $3(4y + 1)(y + 1)$
(f) $(5c - 4)(c + 1)$
(j) $(5r + 2)(r - 2)$

6(a) $(4x + 5)(2x + 3)$
(e) $(2y + 3)^2$
(f) $(2x + 3)(5x - 2)$
(j) $(4x - 5)(2x - 3)$

6. Factor, using the method you like better.

(*a*)	$8x^2 + 22x + 15$	(*f*)	$10x^2 + 11x - 6$
(*b*)	$6c^2 + 5c - 4$	(*g*)	$12s^2 + 10s - 12$
(*c*)	$6r^2 - 5r - 6$	(*h*)	$9a^2 - 24a + 16$
(*d*)	$14t^2 - 27t + 9$	(*i*)	$14t^2 + 29t - 15$
(*e*)	$4y^2 + 12y + 9$	(*j*)	$8x^2 - 22x + 15$

3.6 Sum or Difference of Cubes

Is 3 a factor of 48?
$\frac{48}{3} = 16$
It is!

A simple sum of cubes is $a^3 + b^3$. You might guess that $a + b$ is a factor of $a^3 + b^3$.

We can check by dividing $a + b$ into $a^3 + b^3$.

$$
\begin{array}{r|l}
 & a^2 - ab + b^2 \\
\hline
a + b & a^3 \qquad\qquad + b^3 \\
 & \underline{a^3 + a^2b} \\
 & \quad - a^2b \\
 & \quad \underline{- a^2b - ab^2} \\
 & \qquad\quad ab^2 + b^3 \\
 & \qquad\quad \underline{ab^2 + b^3} \\
 & \qquad\qquad 0
\end{array}
$$

It is a factor!

* Factors of a sum of cubes.

$$\therefore a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

* Factors of a difference of cubes.

Similarly we could show that

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Do you see the pattern?

Example 1

Factor $x^3 - 8$.

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$

SAME SIGN

MULTIPLY x AND 2

CHANGE THE SIGN

SQUARE THE x

* Take the cube root of $8m^3$ and $27n^3$.

Example 2

*Factor $8m^3 + 27n^3$.

$$8m^3 + 27n^3 = (2m)^3 + (3n)^3$$
$$= (2m + 3n)(4m^2 - 6mn + 9n^2)$$

SAME SIGN

SQUARE $3n$

MULTIPLY $2m$ AND $3n$

CHANGE THE SIGN

SQUARE $2m$

Strategy for Factoring Sum or Difference of Cubes

- Take the cube root of the two quantities.
- Set up the binomial factor using the cube roots and the same sign as the given binomial.
- Set up the trinomial factor by:
  - —squaring the first term of the binomial factor,
  - —multiplying the two terms of the binomial and using the opposite sign to the binomial,
  - —squaring the second term of the binomial factor.

Exercise 3.6

Discussion

1. State the cube roots of the two quantities in the following sums or differences of cubes.

(a)	$x^3 + y^3$	(g)	$8e^3 - 125f^3$
(b)	$8r^3 - s^3$	(h)	$64a^3 - b^3$
(c)	$27t^3 + 8u^3$	(i)	$8y^3 - 64x^3$
(d)	$125x^3 - 8y^3$	(j)	$m^3 + 8n^3$
(e)	$8a^3 - b^3$	(k)	$8r^3 - 27s^3$
(f)	$27c^3 + d^3$	(l)	$b^3 + 27c^3$

That is, find "a" and "b".

Written Solutions

2. Factor the following sums of cubes.

(a)	$a^3 + 8b^3$	(d)	$27c^3 + 1$
(b)	$8r^3 + 27s^3$	(e)	$64m^3 + 27$
(c)	$125x^3 + y^3$	(f)	$8t^3 + 1$

3. Factor the following differences of cubes.

(a)	$a^3 - 8b^3$	(d)	$b^3 - 1$
(b)	$x^3 - 64$	(e)	$8m^3 - 27n^3$
(c)	$125e^3 - 1$	(f)	$8e^3 - 125$

4. Factor the following.

(a)	$m^3 - 125$	(d)	$8x^3 - 1$
(b)	$a^3 + 27b^3$	(e)	$27d^3 + 125$
(c)	$8b^3 - 125$	(f)	$125e^3 - 8$

5. Factor the following.

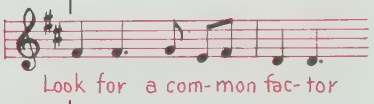
(a)	$2a^3 - 16b^3$	(d)	$32t^3 - 4$
(b)	$3x^3 + 3y^3$	(e)	$7b^3 - 56$
(c)	$3m^3 - 24n^3$	(f)	$5f^3 + 40$

2(a) $(a + 2b)(a^2 - 2ab + 4b^2)$
(d) $(3c + 1)(9c^2 - 3c + 1)$

3(a) $(a - 2b)(a^2 + 2ab + 4b^2)$
(d) $(b - 1)(b^2 + b + 1)$

4(b) $(a + 3b)(a^2 - 3ab + 9b^2)$
(e) $(3d + 5)(9d^2 - 15d + 25)$

5(b) $3(x + y)(x^2 - xy + y^2)$
(e) $7(b - 2)(b^2 + 2b + 4)$



The Remainder Theorem

The remainder theorem is a very useful device for finding the remainder when a polynomial is divided by a binomial.

Remember
Dividend = Divisor × Quotient + Remainder.

Let's begin with a simple division problem.

$$\begin{array}{r|l} & 9 \\ 5 & 49 \\ & 45 \\ \hline & 4 \end{array} \qquad \therefore\ 49 = 5 \times 9 + 4$$

In an algebraic example, the same type of division statement can be made.

$$\begin{array}{r|l} & x + 2 \\ x - 3 & x^2 - x + 7 \\ & x^2 - 3x \\ \hline & 2x + 7 \\ & 2x - 6 \\ \hline & 13 \end{array}$$

DIVISION STATEMENT

$$\therefore\ x^2 - x + 7 = (x - 3)(x + 2) + 13$$

Write the division statement for each of the following problems.

(*a*) $63 \div 8$
(*b*) $72 \div 10$
(*c*) $28 \div 9$
(*d*) $17 \div 4$
(*e*) $38 \div 4$
(*f*) $93 \div 5$
(*g*) $(x^2 + 5x + 5) \div (x + 2)$
(*h*) $(x^2 + x - 11) \div (x - 3)$
(*i*) $(x^2 - 4x - 4) \div (x + 1)$
(*j*) $(x^2 - 7x + 9) \div (x - 5)$
(*k*) $(x^2 + 8x + 3) \div (x + 7)$
(*l*) $(x^2 - 5x + 5) \div (x - 2)$
(*m*) $(x^2 - x - 40) \div (x - 7)$
(*n*) $(x^2 + 4x + 15) \div (x - 3)$
(*o*) $(x^2 + 10x + 25) \div (x + 7)$
(*p*) $(x^2 - 8x - 30) \div (x - 11)$
(*q*) $(x^2 + 5x - 20) \div (x + 8)$
(*r*) $(x^2 - 5x - 35) \div (x - 9)$

When we examine the division statement in the example, we see some rather interesting facts.

$$x^2 - x + 7 = (x - 3)(x + 2) + 13$$

If we substitute $x = 3$, look what happens.

$$3^2 - 3 + 7 = (3 - 3)(3 + 2) + 13$$
$$9 - 3 + 7 = 0 \times 6 + 13$$
$$13 = 13$$

In other words, the value of $x^2 - x + 7$ when $x = 3$ is the value of the remainder when $x^2 - x + 7$ is divided by $x - 3$.

Check this idea with the following statements.

1. (*a*) The value of $x^2 + 5x + 5$ for $x = -2$
   (*b*) The remainder for $(x^2 + 5x + 5) \div (x + 2)$
2. (*a*) The value of $x^2 + x - 11$ for $x = 3$
   (*b*) The remainder for $(x^2 + x - 11) \div (x - 3)$
3. (*a*) The value of $x^2 - 4x - 4$ for $x = -1$
   (*b*) The remainder for $(x^2 - 4x - 4) \div (x + 1)$
4. (*a*) The value of $x^2 - 7x + 9$ when $x = 5$
   (*b*) The remainder for $(x^2 - 7x + 9) \div (x - 5)$
5. (*a*) The value of $x^2 + 8x + 3$ when $x = -7$
   (*b*) The remainder for $(x^2 + 8x + 3) \div (x + 7)$

The value of the remainder when any algebraic expression is divided by $x - a$ is the same as the value of the expression for $x = a$.

1. Find the remainder.
   (*a*) $x^2 + x - 14$ is divided by $x - 3$
   (*b*) $x^3 - 2x^2 - 4x + 3$ is divided by $x - 2$
   (*c*) $x^3 - 3x - 2$ is divided by $x + 2$
   (*d*) $x^3 - x^2 - 5x - 3$ is divided by $x + 3$
   (*e*) $x^3 + x^2 - 4x - 4$ is divided by $x - 1$
   (*f*) $x^3 + 3x^2 - x - 3$ is divided by $x - 2$
2. Find the remainder.
   (*a*) $x^2 - 4x - 5$ is divided by $x + 1$
   (*b*) $x^2 - 6x + 9$ is divided by $x - 3$
   (*c*) $x^2 + x - 12$ is divided by $x + 4$
   (*d*) $x^2 + 4x - 21$ is divided by $x - 3$
   (*e*) $x^3 + 5x^2 + 9x + 9$ is divided by $x + 3$
   (*f*) $x^3 - 6x^2 + 15x - 14$ is divided by $x - 2$

What is your conclusion in each part of Question 2?

The Factor Theorem

When a polynomial is divided by $x - a$, the remainder can be found by simply determining the value of the polynomial when $x = a$. If the remainder is zero, then the binomial $x - a$ is a factor.

Example

Find the remainder: $x^3 + 3x^2 - 4x - 12 \div x + 3$.

Solution

Let $x = -3$ in $x^3 + 3x^2 - 4x - 12$

$$\therefore (-3)^3 + 3(-3)^2 - 4(-3) - 12 = -27 + 27 + 12 - 12$$
$$= 0$$

This means that $x + 3$ is a factor. Division will give another factor.

$$\begin{array}{r} x^2 - 4 \\ x + 3 \overline{)\, x^3 + 3x^2 - 4x - 12} \\ \underline{x^3 + 3x^2} \\ 0 - 4x - 12 \\ \underline{-4x - 12} \\ 0 \end{array}$$

DIFFERENCE OF SQUARES

$$\therefore x^3 + 3x^2 - 4x - 12 = (x + 3)(x^2 - 4)$$
$$= (x + 3)(x + 2)(x - 2)$$

The trick, of course, is determining the value to substitute for the variable in the first place.

Consider $x^2 + x - 6 = (x + 3)(x - 2)$. Notice that both $+3$ and -2 divide exactly into -6. This can be used as a guide in selecting values to substitute.

For instance, for the polynomial $x^3 - 2x^2 - 5x - 6$ the set of values to substitute would be $\{+1, -1, +2, -2, +3, -3, +6, -6\}$ since each of these will divide into -6 with no remainder.

1. What values should be selected to substitute for the variable to test for a factor, in each of the following?
   (*a*) $x^3 - 2x^2 - 5x - 6$
   (*b*) $x^3 + x^2 - 4x - 4$
   (*c*) $x^3 + 7x^2 + 16x + 12$
   (*d*) $x^4 - 13x^2 + 36$
   (*e*) $x^3 + 3x^2 - 18x - 40$
   (*f*) $x^3 + 2x^2 - 11x - 12$
   (*g*) $x^3 - 27$
   (*h*) $x^3 - 10x^2 + 31x - 30$
2. Make the substitutions in Question 1, and find the factors in each case.

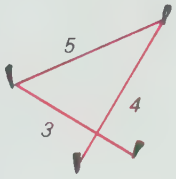
When dividing x^3-27 by $x-3$, don't forget to substitute 0 for the missing terms in x^2 and x.

$x-3 \overline{)\, x^3+0+0-27}$

Pythagoras

"The square of the hypotenuse of a right-angled triangle equals the sum of the squares of the other two sides."

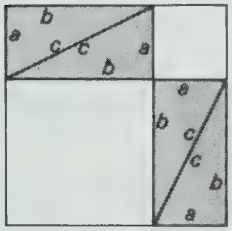
You've seen this before haven't you? It's one of the great theorems from mathematics, so practical it has been in use for thousands of years, and it's still in use today. When you build a cottage and you use a 3-4-5 or 6-8-10 triangle to get a square corner, you will be shaking hands across the centuries with the ancients who used the same triangle in Egypt and Greece.



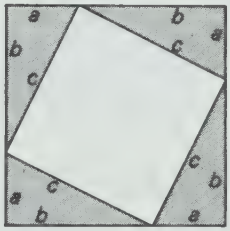
A great number of proofs have been worked out for the Pythagorean Theorem. Here is one you can construct for yourself.

1. Make 8 right triangles the same size. Label the sides.

2. Arrange 4 triangles like this. Draw the outlines of the 2 squares.



3. Arrange the other 4 triangles like this. Draw the outline of the square as shown.



4. Compare the sizes of the *two large squares*. Do they both have sides $(a + b)$ units long? *Are the two areas equal*?

5. Mark the sides of the smaller squares a, b, or c. Color in the 3 squares $a \times a$, $b \times b$, and $c \times c$.

6. Now take away the 8 triangles. What is left must still be equal, or

$$c^2 = a^2 + b^2$$

Unit 4
Uses of Factoring

Now that you have had some practice in factoring expressions, let's see how factoring may be used in handling various types of problems.

4.1 Multiplying and Dividing in Algebra

A *Rational numbers*

Remember how you learned to multiply and divide with *rational numbers*?

1. $$\frac{3}{4} \times \frac{8}{15} = \frac{\cancel{3}^{1}}{\cancel{4}_{1}} \times \frac{\cancel{4}^{1} \times 2}{\cancel{3}_{1} \times 5}$$
$$= \frac{2}{5}$$

3 and 4 are common factors.

2. $$\frac{-7}{8} \div \frac{-35}{24} = \frac{-7}{8} \times \frac{24}{-35}$$
$$= \frac{\cancel{-7}^{1}}{\cancel{8}_{1}} \times \frac{\cancel{8}^{1} \times 3}{\cancel{-7}_{1} \times 5}$$
$$= \frac{3}{5}$$

To divide by a fraction, multiply by its reciprocal.

B *Rational algebraic expressions*

You can use the same method with *rational algebraic expressions*.

Remember to factor first.

1. $\dfrac{x+3}{x+5} \times \dfrac{3x+15}{2x+6} = \dfrac{\cancel{x+3}}{\cancel{x+5}} \times \dfrac{3\cancel{(x+5)}}{2\cancel{(x+3)}}$

$= \dfrac{3}{2}$

2. $\dfrac{x^2+6x+5}{x^2-25} \div \dfrac{x^2+4x+3}{x^2-4x-5}$

"Cancel" the common factors.

$= \dfrac{\cancel{(x+5)}(x+1)}{\cancel{(x-5)}\cancel{(x+5)}} \times \dfrac{\cancel{(x-5)}\cancel{(x+1)}}{(x+3)\cancel{(x+1)}}$

$= \dfrac{x+1}{x+3}$

Exercise 4.1

Discussion

RATIONALS ⇔ FRACTIONS

1. Examine the rationals below and answer these questions for each.
   (i) Can it be reduced to lower terms?
   (ii) If it can, how do you do it?
   (iii) If it cannot, why not?

   (*a*) $\dfrac{4}{5}$ (*d*) $\dfrac{3}{8}$ (*g*) $\dfrac{4-1}{5-2}$

   (*b*) $\dfrac{6}{8}$ (*e*) $\dfrac{4}{14}$ (*h*) $\dfrac{5+3}{5-3}$

   (*c*) $\dfrac{3}{2}$ (*f*) $\dfrac{2+3}{5+3}$ (*i*) $\dfrac{7+3}{7-4}$

2. Examine the rational algebraic expressions and explain why each cannot be reduced.

   (*a*) $\dfrac{x+3}{x-3}$ (*b*) $\dfrac{x^2+5}{x+5}$ (*c*) $\dfrac{3x-5}{3x-2}$

Written Solutions

3. Simplify.

(a) $\frac{3}{5} \times \frac{5}{6}$

(b) $\frac{5}{7} \times \frac{14}{2}$

(c) $\frac{7}{9} \times \frac{18}{21}$

(d) $\frac{-2}{3} \times \frac{-3}{7}$

(e) $\frac{-8}{15} \times \frac{5}{-4}$

(f) $\frac{21}{15} \times \frac{-5}{7}$

4. Simplify.

(a) $\frac{x(x+3)}{(x+1)(x+3)} \times \frac{(x+1)(x-2)}{(x-2)x}$

(b) $\frac{(x+2)(x-7)}{x+3} \times \frac{x+3}{x-7}$

(c) $\frac{(x+2)(x-2)}{x(x+2)} \times \frac{x(x+3)}{x(x-2)}$

(d) $\frac{(4a-3)(a-3)}{(2a-3)(4a-3)} \times \frac{2a-3}{a}$

5. Simplify.

(a) $\frac{4}{7} \div \frac{2}{7}$

(b) $\frac{5}{21} \div \frac{15}{7}$

(c) $\frac{24}{15} \div \frac{-9}{5}$

(d) $\frac{-8}{3} \div \frac{24}{9}$

(e) $\frac{87}{6} \div \frac{58}{3}$

(f) $\frac{-72}{5} \div \frac{16}{25}$

6. Simplify.

(a) $\frac{3x-15}{2x-10} \div \frac{3x^2+9x}{4x^3+12x^2}$

(b) $\frac{x^2-x}{5x-5} \div \frac{x^2y-x^2}{15y-15}$

(c) $\frac{4x^3-8x^2}{6xy-2y} \div \frac{2x^2-4x}{9x^2y-3xy}$

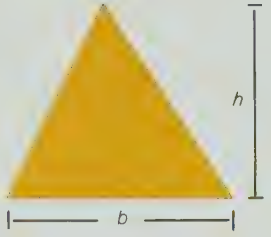
(d) $\frac{12+6a}{15-3a} \div \frac{4-a^2}{25-a^2}$

(e) $\frac{b^2-49}{(b+7)^2} \div \frac{3b-21}{2b+14}$

(f) $\frac{c^2+3c+2}{(c+1)^2} \div \frac{c^2+4c+3}{c^2+5c+6}$

Factor first. Then cancel common factors.

3.(b) 5 (f) -1
4.(b) $(x+2)$
5.(a) 2 (d) -1
6.(a) $2x$ (e) $\frac{2}{3}$
(c) $3x^2$



7. The area of a triangle is calculated using the formula $A = \frac{1}{2}bh$.

   Calculate A for each part.

   (a) $b = 4, h = 7$

   (b) $b = \frac{1}{3}, h = 12$

   (c) $b = \frac{4}{5}, h = \frac{15}{4}$

   (d) $b = \frac{4}{7}, h = \frac{21}{2}$

   (e) $b = \frac{3}{5}, h = \frac{35}{3}$

   (f) $b = \frac{2}{7}, h = 14$

8. Simplify.

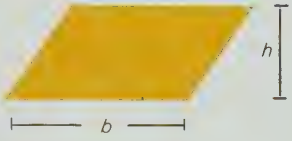
   (a) $\frac{3a - 15}{2a + 3} \times \frac{6a + 9}{5a^2 - 25a}$

   (b) $\frac{3e + 5}{2e + 1} \div \frac{6ef + 10f}{6e^2 + 2e}$

   (c) $\frac{x - 1}{2x^2 - x - 1} \div \frac{6x + 3}{4x^2 + 4x + 1}$

   (d) $\frac{x^2 - 4x + 3}{x^2 - 5x + 6} \times \frac{x^2 - 3x + 2}{x^2 - 2x + 1}$

   (e) $\frac{f^2 - 25}{f^2 - 6f + 5} \div \frac{f^2 + 8f + 15}{f^2 - 9}$



9. The area of a parallelogram is calculated using the formula $A = bh$. Calculate A for the following.

   (a) $b = \frac{3}{7}, h = 14$

   (b) $b = \frac{5}{9}, h = \frac{18}{7}$

   (c) $b = 22, h = \frac{3}{11}$

   (d) $b = \frac{63}{15}, h = \frac{45}{7}$

   (e) $b = 33, h = \frac{2}{11}$

   (f) $b = \frac{17}{3}, h = \frac{33}{51}$

10. Simplify.

    (a) $\frac{d^2 - 25}{d^2 - 10d + 25} \div \frac{d^2 + 4d - 5}{d^2 - 1}$

    (b) $\frac{y^3 - 3y^2}{2y^2 + 7y + 3} \times \frac{2y^2 + y}{y^4 - 3y^3}$

    (c) $\frac{3x + 2}{4x + 3} \times \frac{8x + 6}{12x^2 + 8x}$

    (d) $\frac{c^2 - 16}{3c^3} \div \frac{c^2 - 5c + 4}{3c^3 - 3c^2}$

    (e) $\frac{m^2 - 6m - 7}{(m + 1)^3} \times \frac{m^2 + 2m + 1}{m^2 - 49}$

7.(a) 14 (d) 3

8.(a) $\frac{9}{5a}$ (f) $\frac{1}{m}$

9.(a) 6 (e) 6

10.(a) $\frac{d + 1}{d - 5}$ (e) $\frac{1}{m + 7}$

11. The volume of a cylinder, a square prism, a rectangular prism, or a triangular prism is calculated using the formula $V = Ah$, where A represents the area of the base. Calculate the volume of the prisms indicated below.

(*a*) cylinder, $r = 14$, $h = \frac{3}{4}$

(*b*) square prism, $x = \frac{2}{3}$, $h = \frac{9}{8}$

(*c*) rectangular prism, $l = \frac{13}{3}$, $w = \frac{9}{11}$, $h = \frac{33}{13}$

(*d*) triangular prism, $b = \frac{3}{8}$, $h_1 = 10\frac{2}{3}$, $h_2 = \frac{7}{8}$

(*e*) cylinder, $r = \frac{7}{11}$, $h = \frac{66}{7}$

(*f*) square prism, $x = \frac{3}{5}$, $h = \frac{75}{27}$

(*g*) rectangular prism, $l = \frac{4}{7}$, $w = \frac{14}{9}$, $h = \frac{27}{8}$

(*h*) triangular prism, $b = \frac{6}{7}$, $h_1 = \frac{21}{5}$, $h_2 = 5$

12. Simplify.

(*a*) $\dfrac{4x^2 - 49}{10x^2 + 33x - 7} \div \dfrac{6x^2 - 19x - 7}{3x^2 + x}$

(*b*) $\dfrac{6a^2 + 11a + 3}{10a^2 + 11a - 6} \times \dfrac{10a^2 + a - 2}{6a^2 + 2a}$

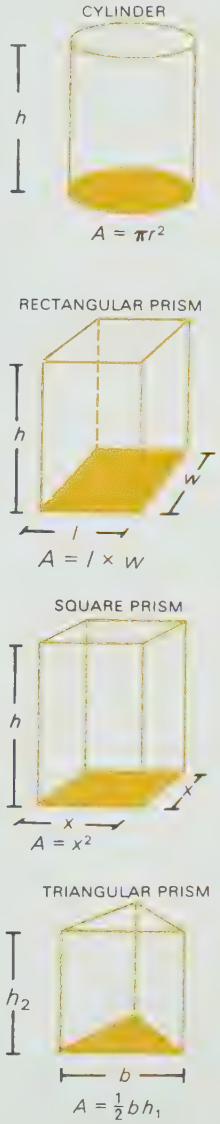
(*c*) $\dfrac{2d^2 - d - 3}{10d^2 - 17d + 3} \div \dfrac{d^2 + 2d + 1}{5d^2 + 4d - 1}$

(*d*) $\dfrac{m^3 + 5m}{m^2 - 25} \times \dfrac{2m^2 - 7m - 15}{2m^2 + 3m}$

(*e*) $\dfrac{2y^2 - 5y - 3}{6y^2 + 7y - 3} \div \dfrac{2y^2 - 17y - 9}{6y^2 + 11y + 3}$

(*f*) $\dfrac{2t^2 + 9t - 5}{3t^2 + 16t + 5} \times \dfrac{3t^2 - 14t - 5}{2t^2 - 11t + 5}$

13. Find an expression for the volume of a cylinder of radius $(m + n)$ and height $(2m - n)$.



11.(a) 462 cubic units
(c) 9
(f) 1
(h) 9

12.(a) $\frac{x}{5x - 1}$ **(d)** $\frac{m^2 + 5}{m + 5}$

4.2 Adding and Subtracting Rational Numbers and Expressions

A The examples will remind you of the important points to remember.

Denominators the same? Just add.

1. $\dfrac{3}{5}+\dfrac{1}{5}=\dfrac{3+1}{5}$

$=\dfrac{4}{5}$

Form equivalent fractions so that the denominators are equal.

2. $\dfrac{4}{7}-\dfrac{1}{14}=\dfrac{8}{14}-\dfrac{1}{14}$

$=\dfrac{7}{14}$

$=\dfrac{1}{2}$

B Rational algebraic expressions are treated in the same manner.

STEPS

1. FACTOR FIRST.
2. CHOOSE LEAST COMMON DENOMINATOR.
3. FORM EQUIVALENT FRACTIONS.
4. ADD.

1. $\dfrac{3}{x+1}-\dfrac{1}{x+1}=\dfrac{3-1}{x+1}$

$=\dfrac{2}{x+1}$

2. $\dfrac{5}{x}+\dfrac{2}{x^2+x}=\dfrac{5}{x}+\dfrac{2}{x(x+1)}$

$=\dfrac{5(x+1)}{x(x+1)}+\dfrac{2}{x(x+1)}$

$=\dfrac{5x+5+2}{x(x+1)}$

$=\dfrac{5x+7}{x(x+1)}$

3. $\dfrac{3}{x^2-25}-\dfrac{2}{x^2-3x-10}=\dfrac{3}{(x-5)(x+5)}-\dfrac{2}{(x-5)(x+2)}$

$=\dfrac{3(x+2)}{(x-5)(x+5)(x+2)}-\dfrac{2(x+5)}{(x-5)(x+5)(x+2)}$

$=\dfrac{3(x+2)-2(x+5)}{(x-5)(x+5)(x+2)}$

$=\dfrac{3x+6-2x-10}{(x-5)(x+5)(x+2)}$

$=\dfrac{x-4}{(x-5)(x+5)(x+2)}$

Exercise 4.2

Discussion

1. Find the smallest number into which 3, 6 and 9 will divide evenly. This number is called the *least common multiple* (LCM).

2. Find the least common multiple for each of the following.
   (*a*) 12, 18, 30
   (*b*) 6, 9, 12
   (*c*) 5, 15, 20
   (*d*) 8, 12, 16
   (*e*) 4, 6, 8
   (*f*) 4, 6, 10

3. Find the least common denominator for each set of fractions.
   (*a*) $\frac{1}{12}, \frac{5}{18}, \frac{7}{30}$
   (*b*) $\frac{1}{3}, \frac{3}{4}, \frac{4}{5}$
   (*c*) $\frac{4}{6}, \frac{1}{8}, \frac{7}{9}$
   (*d*) $\frac{1}{6}, \frac{7}{8}, \frac{3}{4}$
   (*e*) $\frac{1}{3}, \frac{1}{12}, \frac{3}{6}$
   (*f*) $\frac{1}{15}, \frac{2}{3}, \frac{5}{12}$

Written Solutions

4. Find the least common multiple for each of the following.
   (*a*) 6, 12, 15
   (*b*) 5, 15, 20
   (*c*) $x^2 - 25,\ x^2 + 10x + 25$
   (*d*) $x^2 + 4x + 3,\ x^2 + 3x$
   (*e*) $x,\ x^2 - 4,\ x^2 + 2x$
   (*f*) $x^2 - 5x + 6,\ x^2 - 3x,\ x^2 - 4$
   (*g*) $x^2 + 5x - 14,\ x^2 + 8x + 7,\ x^2 - x - 2$
   (*h*) $x^2 - 49,\ x^2 - 6x - 7,\ x^2 - 8x + 7$

5. Determine the least common denominator and then simplify.
   (*a*) $\frac{4}{9} + \frac{5}{12} - \frac{1}{6}$
   (*b*) $\frac{5}{12} - \frac{7}{15} + \frac{2}{6}$
   (*c*) $\frac{3}{x} + \frac{5}{x(x+1)}$
   (*d*) $\frac{3}{4} - \frac{2}{5} + \frac{1}{6}$
   (*e*) $\frac{5}{7} - \frac{4}{15} - \frac{8}{21}$
   (*f*) $\frac{1}{x^2 - 25} - \frac{2}{x^2 - 10x + 25}$
   (*g*) $\frac{7}{x^2 + 3x + 2} + \frac{5}{x^2 - 4}$
   (*h*) $\frac{5}{x+1} - \frac{3}{(x+1)(x+2)}$

$3 = 3$
$6 = 2 \times 3$
$9 = 3 \times 3$
LCM $= 3 \times 3 \times 2$
$= 18$

The LCD is the LCM of the denominators.

4.(a) 60
(d) $x(x+1)(x+3)$
(g) $(x+7)(x+1)(x-2)$

5.(b) $\frac{17}{60}$
(d) $\frac{31}{60}$
(e) $\frac{1}{15}$
(h) $\frac{5x+7}{(x+1)(x+2)}$

6. Simplify.

(a) $\frac{x+2}{4}+\frac{2x-3}{6}-\frac{x+2}{9}$

(b) $\frac{2x-2}{8}+\frac{3x+3}{6}-\frac{3x+5}{4}$

(c) $\frac{3x+2y}{5}-\frac{x-y}{3}+4x$

(d) $\frac{2x-1}{2}+\frac{4x-5}{3}-\frac{x}{6}$

(e) $\frac{5x}{2x+4}+\frac{2x}{4x+8}$

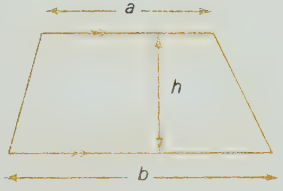
(f) $\frac{x}{x-5}-\frac{5}{x+5}$

(g) $\frac{3x-5}{3x+5}-\frac{3x+5}{3x-5}$

(h) $\frac{x^2}{9x^2-4}-\frac{4}{3x+2}$

(i) $\frac{3x}{2x-3}-\frac{2x}{2x+3}+\frac{3}{4x^2-9}$

(j) $\frac{5x}{4x+1}+\frac{2x}{4x-1}-\frac{3}{16x^2-1}$



7. The area of a trapezoid is calculated using the formula $A=\frac{1}{2}(a+b)h$. Calculate A, for each of the following.

(a) $a=2,\ b=3,\ h=6$

(b) $a=\frac{1}{2},\ b=\frac{1}{3},\ h=\frac{3}{5}$

(c) $a=\frac{3}{5},\ b=\frac{2}{3},\ h=\frac{5}{19}$

(d) $a=\frac{8}{9},\ b=\frac{4}{5},\ h=\frac{15}{19}$

(e) $a=\frac{5}{8},\ b=\frac{5}{6},\ h=\frac{12}{7}$

(f) $a=\frac{7}{12},\ b=\frac{13}{15},\ h=\frac{20}{29}$

(g) $a=\frac{3}{4},\ b=\frac{1}{6},\ h=\frac{8}{11}$

(h) $a=\frac{5}{6},\ b=\frac{2}{9},\ h=\frac{18}{19}$

8. Simplify.

(a) $\frac{7a}{3}-\frac{2a}{5}+\frac{3a}{10}$

(b) $\frac{2}{y-4}-\frac{y}{y+4}-\frac{y+2}{y^2-16}$

(c) $\frac{x+3}{x^2+3x}-\frac{x-3}{x^2-9}$

6.(b) -1

(e) $\frac{3x}{x+2}$

(f) $\frac{x^2+25}{(x-5)(x+5)}$

(j) $\frac{28x^2-3x-3}{(4x+1)(4x-1)}$

7.(a) 15 (f) $\frac{1}{2}$

(c) $\frac{1}{6}$ (h) $\frac{1}{2}$

(*d*) $\dfrac{3m-9}{7} - \dfrac{2m+3}{5} - \dfrac{m-2}{2}$

(*e*) $\dfrac{3}{x-1} - \dfrac{6}{(x+1)(x-1)}$

(*f*) $\dfrac{3}{a+b} - \dfrac{5}{a-b} + \dfrac{2}{a+b}$

(*g*) $\dfrac{r+1}{2} - \dfrac{2r-1}{3} + \dfrac{5r}{4}$

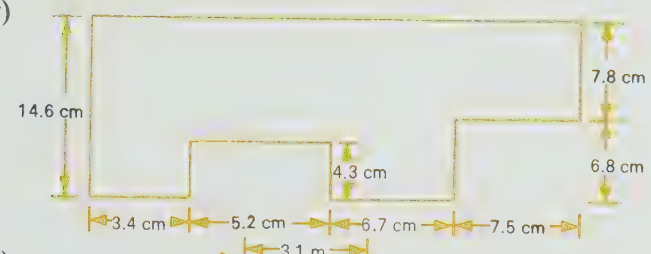
(*h*) $\dfrac{4}{x-1} - \dfrac{4}{x+1}$

(*i*) $\dfrac{x^2+3x+2}{x^2-1} - \dfrac{2x}{x+1}$

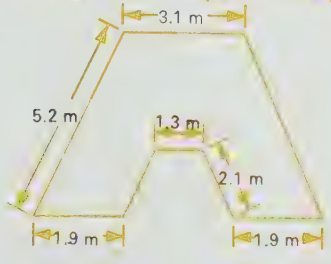
(*j*) $\dfrac{1}{y^2-9y+20} + \dfrac{1}{y^2-11y+30}$

9. Calculate the perimeter of each of the following.

(*a*)



(*b*)



10. The area of an annulus or ring is calculated using the formula $A = \pi(R^2 - r^2)$. Calculate A, for each of the following.

(*a*) $R = \dfrac{2}{3}, r = \dfrac{1}{2}$ (*c*) $R = \dfrac{4}{5}, r = \dfrac{1}{3}$

(*b*) $R = \dfrac{5}{6}, r = \dfrac{2}{3}$ (*d*) $R = \dfrac{7}{8}, r = \dfrac{1}{4}$

How could you alter the formula $A = \pi(R^2 - r^2)$ in order to avoid squaring R and r?

10.

8.(d) $\dfrac{-33m-62}{70}$

(g) $\dfrac{13r+10}{12}$

(j) $\dfrac{2}{(y-4)(y-6)}$

9.(a) 83.4 cm

10.(b) $\frac{11}{14}$ (c) $\frac{374}{225}$

SECOND DEGREE

$x^2 + 3x + 2 = 0$

4.3 Solving Quadratic Equations by Factoring

The solution of second degree or quadratic equations is easy if factoring is possible.

1. Solve $x^2 + 3x + 2 = 0$.

   Solution $(x + 2)(x + 1) = 0$

   Factor first. One of them must be zero if the product is zero.

   $\therefore x + 2 = 0$ or $x + 1 = 0$
   $\therefore x = -2$ or $x = -1$
   $\therefore$ solution is $x = -1$ or -2.

   "Roots" of the equation.

2. Solve $x^2 - 5x + 2 = -2$.

   Arrange all terms on the left. Then factor.

   Solution $x^2 - 5x + 4 = 0$
   $(x - 4)(x - 1) = 0$
   $\therefore x - 4 = 0$ or $x - 1 = 0$
   $\therefore x = 4$ or $x = 1$
   $\therefore$ solution is $x = 1$ or 4.

3. Find the roots of $x^2 - 3x + 5 = 5$. Verify.

   Solution $x^2 - 3x = 0$
   $\therefore x(x - 3) = 0$
   $\therefore x = 0$ or $x - 3 = 0$
   $\therefore x = 0$ or $x = 3$
   $\therefore$ the roots are 0 and 3.

Verification		
	LS	RS
$x = 0$	$x^2 - 3x + 5$ $= 0 - 0 + 5$ $= 5$	5
$x = 3$	$x^2 - 3x + 5$ $= 9 - 9 + 5$ $= 5$	5

Exercise 4.3

Discussion

1. State the roots of each of the following.

   (a) $(x + 2)(x - 1) = 0$
   (b) $(y - 2)(y - 3) = 0$
   (c) $(x - 5)(x - 2) = 0$
   (d) $(m + 2)(m + 5) = 0$
   (e) $(t + 7)(t - 5) = 0$
   (f) $x(x - 2) = 0$
   (g) $a(a + 7) = 0$
   (h) $(r + 11)(r - 3) = 0$
   (i) $y\left(y + \frac{4}{3}\right) = 0$
   (j) $(c - 3)\left(c + \frac{7}{3}\right) = 0$

Written Solutions

2. Solve.

(a) $m^2 - 6m + 8 = 0$	(f) $y^2 - 3y + 2 = 0$
(b) $h^2 - 3h - 4 = 0$	(g) $y^2 - 6y = 0$
(c) $c^2 + 2c - 8 = 0$	(h) $p^2 - p - 20 = 0$
(d) $a^2 + 8a - 65 = 0$	(i) $x^2 + 7x + 10 = 0$
(e) $x^2 + 7x - 30 = 0$	(j) $y^2 - 81 = 0$

3. Find the roots.

(a) $a^2 - 8a + 16 = 25$	(f) $m^2 - 2m = 15$
(b) $d^2 + 12d = 64$	(g) $x^2 = 6x - 8$
(c) $x^2 - 8x = 33$	(h) $x(x - 1) = 12$
(d) $c^2 = 8c - 15$	(i) $f(f + 12) = 64$
(e) $y^2 + 7y = 30$	(j) $(a - 2)(a + 2) = 21$

4. Solve and verify.

(a) $d^2 + 11d + 30 = 0$	(d) $r^2 = 5r$
(b) $63 = 2y + y^2$	(e) $6 - x - x^2 = 0$
(c) $c^2 - 8c = 0$	(f) $s^2 - 3s - 4 = 0$

5. One side of a rectangular sheet of tin is 6 cm more than the other. The area of the sheet is 7 cm^2. Calculate the dimensions of the sheet of tin.

6. Find two consecutive numbers such that their product is 72.

7. The area of a triangular plate is 96 cm^2. If the plate is as shown here, find the base and the altitude.

8. The area of the figure shown here is 45 m^2. Find the width of the arms.

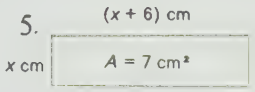
9. Find two consecutive positive even numbers such that their product is 168.

10. A rectangular parking area had an area of 799 m^2. If the length was 30 m longer than the width, find the dimensions.

11. The area of the shaded region is 1440 cm^2. Find the dimensions of the rectangular centre section.

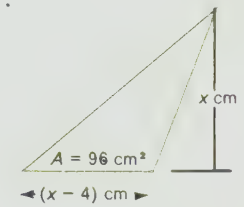
12. Find two consecutive odd numbers such that their product is 399.

5.



6. Try x and $x+1$. (And get two answers.)

7.



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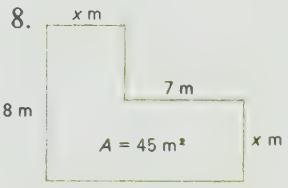
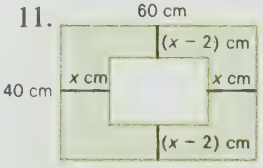


11.



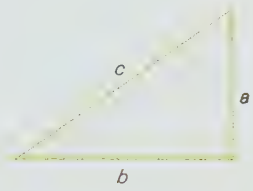
2.(a) 2, 4 **(f) 1, 2**
(e) −10, 3 **(j) 9, −9**
3.(b) −16, 4 **(g) 2, 4**
(d) 3, 5 **(i) −16, 4**
4.(a) −6, −5 **(d) 0, 5**
5. 1 cm × 7 cm
7. base 12 cm
alt. 16 cm
8. 3 m
9. 12 and 14
10. 17 m × 47 m
11. 40 cm × 24 cm

Unit 5
Working with Radicals

5.1 Radicals Exist!

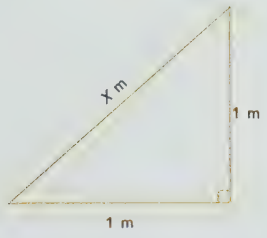
Simply stated, radicals are expressions involving root signs. Such expressions occur frequently in certain areas of mathematics.

Some simple geometric problems will indicate the fact that radical numbers, such as $\sqrt{7}$, $\sqrt{3}$, $\sqrt{2}$, are part of life and so we must be able to work with them.



Remember Pythagoras?
$a^2 + b^2 = c^2$
(See page 53)

1. Determine the length of the hypotenuse of the right-angled triangle shown.



Solution

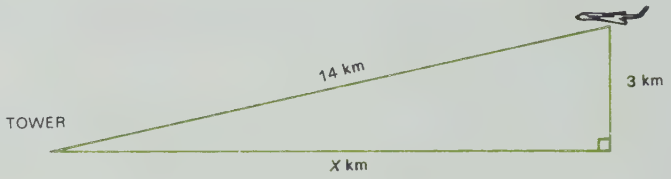
Using the Law of Pythagoras:

$$x^2 = 1^2 + 1^2$$
$$x^2 = 1 + 1$$
$$= 2$$
$$\therefore x = \sqrt{2}$$
$$\doteq 1.414 \quad \text{(Square root table)}.$$

This is an irrational number: a never-ending, never-repeating decimal.

The length of the hypotenuse is approximately 1.414 m, and radicals such as $\sqrt{2}$ can represent the length of a segment.

2. The image of a plane is detected on the radar screen of a control tower and is estimated to be 14 km away from the control tower. If the plane is flying at an altitude of 3 km, how far horizontally is the plane from the tower?



Solution

Using the Law of Pythagoras:

$$\begin{aligned} 14^2 &= x^2 + 3^2 \\ 196 &= x^2 + 9 \\ x^2 &= 196 - 9 \\ x &= \sqrt{187} \\ \therefore x &\doteq 13.7 \end{aligned}$$ (Square root table)

$\therefore$ the plane is approximately 13.7 km horizontally from the control tower.

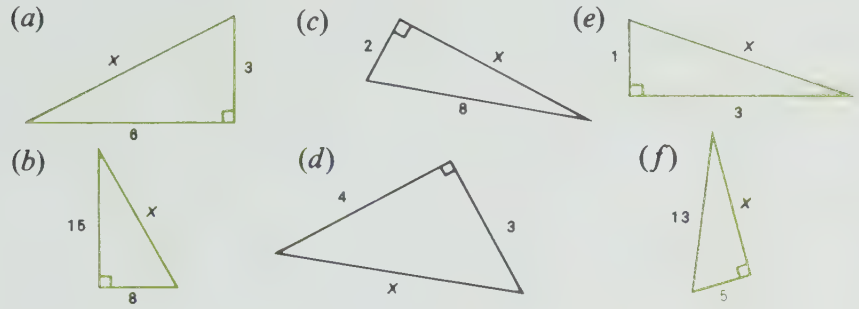
Exercise 5.1

Written Solutions

1. Which of the following can be the lengths of the sides of a right-angled triangle?
   (*a*) 3, 4, 5
   (*b*) 1, $\sqrt{3}$, 2
   (*c*) 2, 3, 4
   (*d*) 5, 12, 13
   (*e*) 5, 6, 8
   (*f*) 1, 1, $\sqrt{2}$
   (*g*) 5, 7, 9
   (*h*) 8, 15, 17
2. Which of the sides in Question 1 represent the hypotenuse?
3. Locate $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$ on a number line, as accurately as you can.
4. Find, to the nearest metre, the distance that the foot of a 10 m ladder will be from a wall if it reaches 8 m up the wall.

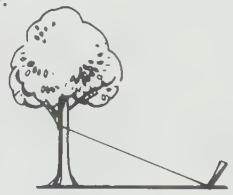
1.(a) (h)
2.(b) 2
(f) $\sqrt{2}$
4. 6 m

5. Calculate the missing lengths in the diagrams (to two decimal places).

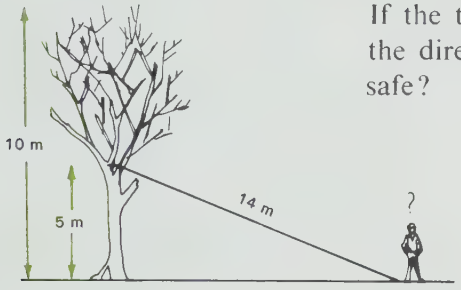


6. To support a recently planted tree, a boy took a piece of rope 4 m long and tied it to the tree 2 m above the ground. He then drove a stake into the ground 3 m from the tree. If a total of 0.5 m of rope is needed to tie both ends, will the rope be long enough?

6.



7.



If the tree shown is toppled in the direction of the man, is he safe?

5.(a) 6.71
(b) 17.00

6. No

5.2 Operations with Radicals, Part 1

A *Multiplication*

Evaluate $\sqrt{3} \times \sqrt{5}$.
From square root tables,

$$\sqrt{3} \times \sqrt{5} \doteq 1.732 \times 2.236$$
$$\doteq 3.873$$

Also from tables,

$$\sqrt{15} \doteq 3.873$$

It appears that $\sqrt{3} \times \sqrt{5} = \sqrt{15}$.

Before you turn the page – what is the rule?

The rule for multiplication of radicals is

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

Using this rule, $\sqrt{2} \times \sqrt{6} = \sqrt{12}$
and $\sqrt{5} \times \sqrt{7} = \sqrt{35}$.

1. Multiply the following radicals.

(*a*)	$\sqrt{3} \times \sqrt{2}$	(*d*)	$\sqrt{10} \times \sqrt{7}$	(*g*)	$\sqrt{12} \times \sqrt{2}$
(*b*)	$\sqrt{7} \times \sqrt{3}$	(*e*)	$\sqrt{3} \times \sqrt{12}$	(*h*)	$\sqrt{5} \times \sqrt{11}$
(*c*)	$\sqrt{5} \times \sqrt{8}$	(*f*)	$\sqrt{6} \times \sqrt{7}$	(*i*)	$\sqrt{13} \times \sqrt{2}$

ENTIRE RADICAL $\sqrt{13}$
MIXED RADICAL $2\sqrt{7}$

B Sometimes radicals appear in a mixed form as in $2\sqrt{3}$ and $5\sqrt{7}$.

Then $2\sqrt{3} \times 5\sqrt{7}$ means

$$2 \times \sqrt{3} \times 5 \times \sqrt{7} = 2 \times 5 \times \sqrt{3} \times \sqrt{7}$$
$$= 10\sqrt{21}$$

MULTIPLY THE WHOLE NUMBERS

$$\therefore 2\sqrt{3} \times 5\sqrt{7} = 10\sqrt{21}$$

MULTIPLY THE RADICALS

$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$

1. State the product.

(*a*)	$\sqrt{2} \times \sqrt{7}$	(*e*)	$\sqrt{3} \times \sqrt{13}$
(*b*)	$\sqrt{3} \times \sqrt{10}$	(*f*)	$\sqrt{17} \times \sqrt{3}$
(*c*)	$\sqrt{2} \times \sqrt{11}$	(*g*)	$\sqrt{5} \times \sqrt{5}$
(*d*)	$\sqrt{7} \times \sqrt{5}$	(*h*)	$\sqrt{7} \times \sqrt{7}$

Express radical answers in lowest terms.
$\sqrt{48} = 4\sqrt{3}$

2. State the product.

(*a*)	$3\sqrt{3} \times 2\sqrt{5}$	(*e*)	$4\sqrt{3} \times \sqrt{11}$
(*b*)	$4\sqrt{7} \times 3\sqrt{11}$	(*f*)	$\sqrt{7} \times 3\sqrt{2}$
(*c*)	$6\sqrt{2} \times 2\sqrt{3}$	(*g*)	$5\sqrt{2} \times 3\sqrt{2}$
(*d*)	$5\sqrt{5} \times \sqrt{7}$	(*h*)	$4\sqrt{7} \times 3\sqrt{7}$

C The rule $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is very helpful in evaluating radicals if we reverse it.

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

For example,

$$\sqrt{500} = \sqrt{100} \times \sqrt{5}$$
$$* = 10\sqrt{5}$$
$$\doteq 10 \times 2.236$$
$$= 22.36$$

Express in lowest terms. Then evaluate.

1. Evaluate the following using your tables.

(*a*) $\sqrt{300}$ (*c*) $\sqrt{250}$ (*e*) $\sqrt{363}$

(*b*) $\sqrt{1000}$ (*d*) $\sqrt{243}$ (*f*) $\sqrt{630}$

D *Division*

Evaluate $\dfrac{\sqrt{15}}{\sqrt{3}}$.

1. Using square root tables,

$$\frac{\sqrt{15}}{\sqrt{3}} \doteq \frac{3.873}{1.732}$$
$$= 2.236$$

2. Factoring makes it easier to divide.

$$\frac{\sqrt{15}}{\sqrt{3}} = \frac{\sqrt{5} \times \sqrt{3}}{\sqrt{3}}$$
$$= \sqrt{5}$$
$$\doteq 2.236$$

3. Notice also that $\dfrac{\sqrt{15}}{\sqrt{3}} = \sqrt{\dfrac{15}{3}} = \sqrt{5}$.

This makes division still easier.
Can you see the rule?

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

4. Using the rule, simplify $\dfrac{6\sqrt{30}}{2\sqrt{5}}$.

$$\frac{6\sqrt{30}}{2\sqrt{5}} = \frac{6}{2}\sqrt{\frac{30}{5}}$$
DIVIDE WHOLE NUMBERS
$$= 3\sqrt{6}$$
DIVIDE RADICALS

$$\frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c}\sqrt{\frac{b}{d}}$$

Try these.

1. State the quotient.

(*a*) $\sqrt{14} \div \sqrt{2}$ (*d*) $\sqrt{84} \div \sqrt{21}$

(*b*) $\sqrt{20} \div \sqrt{10}$ (*e*) $\sqrt{45} \div \sqrt{5}$

(*c*) $\sqrt{10} \div \sqrt{2}$ (*f*) $\sqrt{28} \div \sqrt{7}$

2. State the quotient.

(*a*) $2\sqrt{15} \div \sqrt{5}$ (*d*) $14\sqrt{34} \div 7\sqrt{17}$

(*b*) $6\sqrt{18} \div 3\sqrt{2}$ (*e*) $5\sqrt{90} \div 10\sqrt{15}$

(*c*) $8\sqrt{20} \div 4\sqrt{10}$ (*f*) $6\sqrt{33} \div 3\sqrt{3}$

Exercise 5.2

Written Solutions

1. Evaluate using the tables.

(a) $\sqrt{14} \times \sqrt{3}$
(b) $\sqrt{3} \times \sqrt{2} \times \sqrt{5}$
(c) $\sqrt{12} \times \sqrt{7}$
(d) $4\sqrt{3} \times 2\sqrt{5}$
(e) $5\sqrt{7} \times 3\sqrt{2}$
(f) $2\sqrt{7} \times 3\sqrt{2} \times 10\sqrt{5}$
(g) $3\sqrt{15} \times 2\sqrt{2}$
(h) $5\sqrt{7} \times 3\sqrt{10}$
(i) $7\sqrt{3} \times \sqrt{5}$
(j) $\sqrt{5} \times 3\sqrt{7}$

2. Multiply.

(a) $\sqrt{\frac{3}{4}} \times \sqrt{\frac{2}{6}}$
(b) $\sqrt{\frac{7}{9}} \times \sqrt{\frac{12}{21}}$
(c) $\sqrt{\frac{4}{5}} \times \sqrt{\frac{15}{2}}$
(d) $\sqrt{\frac{15}{17}} \times \sqrt{\frac{34}{3}}$
(e) $\sqrt{\frac{3}{7}} \times \sqrt{\frac{7}{9}}$
(f) $\sqrt{\frac{2}{5}} \times \sqrt{\frac{3}{7}}$

3. Evaluate using the tables.

(a) $\sqrt{14} \div \sqrt{2}$
(b) $\sqrt{21} \div \sqrt{7}$
(c) $2\sqrt{10} \div \sqrt{5}$
(d) $\sqrt{15} \div (3\sqrt{3})$
(e) $8\sqrt{18} \div (4\sqrt{3})$
(f) $48\sqrt{35} \div (16\sqrt{7})$
(g) $3\sqrt{75} \div \sqrt{15}$
(h) $5\sqrt{54} \div (10\sqrt{18})$
(i) $\sqrt{\frac{6}{7}} \div \sqrt{\frac{3}{14}}$
(j) $\sqrt{\frac{3}{4}} \div \sqrt{\frac{1}{8}}$

4. Divide.

(a) $\sqrt{\frac{4}{5}} \div \sqrt{\frac{5}{16}}$
(b) $\sqrt{\frac{7}{8}} \div \sqrt{\frac{7}{32}}$
(c) $15\sqrt{\frac{24}{9}} \div \left(3\sqrt{\frac{1}{6}}\right)$
(d) $27\sqrt{\frac{9}{4}} \div \left(54\sqrt{\frac{3}{12}}\right)$
(e) $3\sqrt{\frac{7}{8}} \div \sqrt{\frac{2}{7}}$
(f) $14\sqrt{\frac{1}{10}} \div \left(7\sqrt{10}\right)$

Memorize:

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$
- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- $\frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c}\sqrt{\frac{b}{d}}$

1.(a) 6.481 (f) 502.02
(c) 9.165 (h) 125.505
2.(c) $\sqrt{6}$ (d) $\sqrt{10}$
3.(b) 1.732 (g) 6.708
(e) 4.898 (j) 2.449
4.(b) 2 (c) 20

5.3 Operations with Radicals, Part 2

A *Addition and Subtraction*

You can add or subtract only like radicals.

You can add $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$

But you cannot add $3\sqrt{2} + 5\sqrt{3}$

B Sometimes quantities can be changed so they form like radicals.

Example

Add $\sqrt{98} + \sqrt{32}$.

Solution

$$\begin{aligned}\sqrt{98} + \sqrt{32} &= \sqrt{49 \times 2} + \sqrt{16 \times 2} \\ &= \sqrt{49}\sqrt{2} + \sqrt{16}\sqrt{2} \\ &= 7\sqrt{2} + 4\sqrt{2} \\ &= 11\sqrt{2}\end{aligned}$$

By turning the "entire" radicals $\sqrt{98}$ and $\sqrt{32}$ into the "mixed" radicals $7\sqrt{2}$ and $4\sqrt{2}$ the addition can be performed.

The radicals are the same.

The radicals are not the same.

They sure dont look alike!

But they are!

Exercise 5.3

Discussion

1. Simplify.
   (*a*) $3\sqrt{2} + \sqrt{2}$ (*d*) $3\sqrt{7} + \sqrt{7}$
   (*b*) $5\sqrt{3} - 2\sqrt{3}$ (*e*) $-2\sqrt{6} + 5\sqrt{6}$
   (*c*) $7\sqrt{5} - 5\sqrt{5}$ (*f*) $4\sqrt{3} - 5\sqrt{3}$

2. Change the entire radical to a mixed radical.
   (*a*) $\sqrt{8}$ (*d*) $\sqrt{75}$ (*g*) $\sqrt{200}$
   (*b*) $\sqrt{27}$ (*e*) $\sqrt{18}$ (*h*) $\sqrt{162}$
   (*c*) $\sqrt{48}$ (*f*) $\sqrt{32}$ (*i*) $\sqrt{54}$

3. Change the mixed radical to an entire radical.
   (*a*) $2\sqrt{3}$ (*d*) $3\sqrt{6}$ (*g*) $6\sqrt{5}$
   (*b*) $3\sqrt{5}$ (*e*) $2\sqrt{7}$ (*h*) $3\sqrt{6}$
   (*c*) $7\sqrt{2}$ (*f*) $5\sqrt{2}$ (*i*) $9\sqrt{5}$

Written Solutions

4. Express in lowest terms.

(a) $\sqrt{50}$	(f) $7\sqrt{48}$	(k) $7\sqrt{45}$
(b) $\sqrt{72}$	(g) $3\sqrt{8}$	(l) $3\sqrt{80}$
(c) $2\sqrt{27}$	(h) $2\sqrt{12}$	(m) $5\sqrt{128}$
(d) $3\sqrt{18}$	(i) $3\sqrt{75}$	(n) $6\sqrt{125}$
(e) $5\sqrt{32}$	(j) $8\sqrt{64}$	(o) $9\sqrt{121}$

5. Simplify.

(a) $\sqrt{18} + \sqrt{50} - \sqrt{72}$
(b) $\sqrt{75} - \sqrt{48} + \sqrt{27}$
(c) $3\sqrt{18} + 2\sqrt{2} - 5\sqrt{8}$
(d) $3\sqrt{98} + 4\sqrt{32} - 2\sqrt{50}$
(e) $5\sqrt{125} + 3\sqrt{45} - 7\sqrt{80}$
(f) $2\sqrt{12} - 3\sqrt{27} + 4\sqrt{8}$
(g) $2\sqrt{24} - 5\sqrt{54} + 3\sqrt{150}$
(h) $5\sqrt{75} + 3\sqrt{12} - 5\sqrt{18}$
(i) $7\sqrt{63} - 4\sqrt{175} - 5\sqrt{28}$
(j) $-9\sqrt{44} + \sqrt{99} - 4\sqrt{176}$

Watch out!

6. Express as a "mixed" radical.

(a) $\sqrt{a^2b}$	(f) $\sqrt{c^5d^2}$	(k) $\sqrt{x^5y^3}$
(b) $\sqrt{x^2y^3}$	(g) $\sqrt{a^2b^3c^3}$	(l) $\sqrt{a^4b^5}$
(c) $\sqrt{m^3}$	(h) $\sqrt{m^3n^5}$	(m) $\sqrt{b^4c^5}$
(d) $\sqrt{p^3q^2}$	(i) $\sqrt{b^7d^6}$	(n) $\sqrt{p^2q^7}$
(e) $\sqrt{r^3s^3}$	(j) $\sqrt{m^6n^5}$	(o) $\sqrt{m^6n^7}$

7. Simplify.

(a) $3\sqrt{a^3b^2} + ab\sqrt{a} - a\sqrt{ab^2}$
(b) $2m^2n^2\sqrt{m^3n} - m\sqrt{m^5n^5}$
(c) $7p^7q^5\sqrt{p^2q^3} - 3p^3q\sqrt{p^{10}q^{11}}$
(d) $5\sqrt{a^6b^5} - 3a^2b\sqrt{a^2b^3}$
(e) $7\sqrt{x^9y^7} - 3x^2y\sqrt{x^5y^5}$
(f) $-3\sqrt{c^5d^3} + 2c\sqrt{c^3d^3}$

4.(a) $5\sqrt{2}$ (b) $6\sqrt{2}$ (g) $6\sqrt{2}$ (h) $4\sqrt{3}$ (n) $30\sqrt{5}$ (o) 99

5.(c) $\sqrt{2}$ (d) $27\sqrt{2}$ (g) $4\sqrt{6}$ (h) $31\sqrt{3} - 15\sqrt{2}$

6.(a) $a\sqrt{b}$ (d) $pq\sqrt{p}$ (g) $abc\sqrt{bc}$ (j) $m^3n^2\sqrt{n}$ (m) $b^2c^2\sqrt{c}$

7.(a) $3ab\sqrt{a}$ (d) $2a^3b^2\sqrt{b}$

5.4 Rationalizing

A An interesting result occurs when two like radicals are multiplied.

1. $3\sqrt{5} \times \sqrt{5} = 3\sqrt{25} = 15$
2. $2\sqrt{7} \times 3\sqrt{7} = 6\sqrt{49} = 42$

The result is a rational number!

It always works!
$a\sqrt{b} \times \sqrt{b}$
$= a\sqrt{b^2}$
$= a\,b$

B Another interesting result involves the sum and difference of the same radicals.

1. $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$
$= \sqrt{9} - \sqrt{15} + \sqrt{15} - \sqrt{25}$
$= \sqrt{9} \qquad\qquad - \sqrt{25}$
$= 3 - 5$
$= -2$

Remember FOIL?

2. $(3\sqrt{2} + 2\sqrt{5})(3\sqrt{2} - 2\sqrt{5})$
$= 9\sqrt{4} - 6\sqrt{10} + 6\sqrt{10} - 4\sqrt{25}$
$= 18 \qquad\qquad\quad - 20$
$= -2$

3. $(2\sqrt{7} - 3\sqrt{3})(2\sqrt{7} + 3\sqrt{3})$
$= 4\sqrt{49} + 6\sqrt{21} - 6\sqrt{21} - 9\sqrt{9}$
$= 28 \qquad\qquad\quad - 27$
$= +1$

Compare to the difference of squares.

Again the result is rational! These binomial radicals are called conjugate radicals. Thus $\sqrt{3} + \sqrt{5}$ and $\sqrt{3} - \sqrt{5}$ are conjugate radicals and the product of conjugate radicals is always rational.

Exercise 5.4

Discussion

1. Simplify.

(*a*) $3\sqrt{5} \times \sqrt{5}$
(*b*) $2\sqrt{7} \times \sqrt{7}$
(*c*) $-5\sqrt{2} \times \sqrt{2}$
(*d*) $4\sqrt{6} \times 2\sqrt{6}$
(*e*) $5\sqrt{11} \times (-\sqrt{11})$
(*f*) $3\sqrt{13} \times 2\sqrt{13}$
(*g*) $-5\sqrt{3} \times \sqrt{3}$
(*h*) $4\sqrt{5} \times 2\sqrt{5}$

$-4\sqrt{6} \times \sqrt{6}$
$= -4 \times 6$
$= -24$

2. What must you multiply each by in order to rationalize?

(a) $-4\sqrt{6}$
(b) $\sqrt{3}+\sqrt{2}$
(c) $15\sqrt{3}$
(d) $\sqrt{6}-\sqrt{3}$
(e) $14\sqrt{7}$
(f) $\sqrt{21}-3$
(g) $3\sqrt{2}+\sqrt{5}$
(h) $5\sqrt{7}-\sqrt{6}$
(i) $3-\sqrt{15}$
(j) $7\sqrt{2}+\sqrt{3}$

Written Solutions

3. Simplify.

(a) $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})$
(b) $(\sqrt{7}+2)(\sqrt{7}-2)$
(c) $(5-\sqrt{3})(5+\sqrt{3})$
(d) $(\sqrt{9}-\sqrt{7})(\sqrt{9}+\sqrt{7})$
(e) $(\sqrt{8}-\sqrt{12})(\sqrt{8}+\sqrt{12})$
(f) $(5+\sqrt{7})(5-\sqrt{7})$

4. Simplify.

(a) $(4\sqrt{3}+2\sqrt{7})(4\sqrt{3}-2\sqrt{7})$
(b) $(\sqrt{5}-3\sqrt{2})(\sqrt{5}+3\sqrt{2})$
(c) $(\sqrt{15}-2\sqrt{3})(\sqrt{15}+2\sqrt{3})$
(d) $(\sqrt{7}+3\sqrt{3})(\sqrt{7}-3\sqrt{3})$
(e) $(3-\sqrt{21})(3+\sqrt{21})$
(f) $(5+\sqrt{7})(5-\sqrt{7})$
(g) $(\sqrt{15}-3)(\sqrt{15}+3)$
(h) $(2\sqrt{6}-3\sqrt{5})(2\sqrt{6}+3\sqrt{5})$
(i) $(5\sqrt{7}+2\sqrt{3})(5\sqrt{7}-2\sqrt{3})$
(j) $(5-\sqrt{17})(5+\sqrt{17})$
(k) $(9\sqrt{2}+3\sqrt{3})(9\sqrt{2}-3\sqrt{3})$
(l) $(7\sqrt{3}+\sqrt{5})(7\sqrt{3}-\sqrt{5})$
(m) $(\sqrt{17}+5)(\sqrt{17}-5)$
(n) $(3\sqrt{6}-4\sqrt{2})(3\sqrt{6}+4\sqrt{2})$
(o) $(2\sqrt{11}-3)(2\sqrt{11}+3)$
(p) $(5+2\sqrt{7})(5-2\sqrt{7})$
(q) $(6+3\sqrt{5})(6-3\sqrt{5})$
(r) $(3\sqrt{11}-2\sqrt{7})(3\sqrt{11}+2\sqrt{7})$

5. Using the table of square roots on page 468, evaluate $\frac{3}{\sqrt{15}}$.

3.(a) 3 (f) 18
(c) 22
4.(a) 20 (h) −21
(b) −13 (m) −8
(g) 6 (n) 22

6. Note that $\dfrac{3}{\sqrt{15}} = \dfrac{3}{\sqrt{15}} \times \dfrac{\sqrt{15}}{\sqrt{15}}$

$$= \frac{3\sqrt{15}}{\sqrt{15}\sqrt{15}}$$

$$= \frac{3\sqrt{15}}{15}$$

$$= \frac{\sqrt{15}}{5}$$

Use the tables to evaluate $\dfrac{\sqrt{15}}{5}$.

By multiplying the denominator by a like radical, we are able to produce a rational denominator. This is called *rationalizing the denominator*.

7. Rationalize the denominator and simplify.

(*a*) $\dfrac{3}{\sqrt{6}}$ (*d*) $\dfrac{42\sqrt{3}}{3\sqrt{7}}$ (*g*) $\dfrac{9\sqrt{5}}{3\sqrt{6}}$

(*b*) $\dfrac{5\sqrt{3}}{\sqrt{2}}$ (*e*) $\dfrac{5\sqrt{5}}{2\sqrt{3}}$ (*h*) $\dfrac{99\sqrt{2}}{3\sqrt{11}}$

(*c*) $\dfrac{27\sqrt{2}}{\sqrt{3}}$ (*f*) $\dfrac{15\sqrt{7}}{\sqrt{3}}$

8. Using the table of square roots, evaluate $\dfrac{7}{\sqrt{5} - \sqrt{2}}$.

9. Notice that

$$\frac{7}{\sqrt{5} - \sqrt{2}} = \frac{7}{\sqrt{5} - \sqrt{2}} \times \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}}$$

$$= \frac{7(\sqrt{5} + \sqrt{2})}{\sqrt{25} - \sqrt{4}}$$

$$= \frac{7(\sqrt{5} + \sqrt{2})}{5 - 2}$$

$$= \frac{7(\sqrt{5} + \sqrt{2})}{3}$$

Evaluate using the tables, and compare your results with those of Question 8. By multiplying the denominator by its conjugate radical we are able to produce a rational denominator. This is again called rationalizing the denominator.

$\frac{\sqrt{15}}{\sqrt{15}} = 1$

Question 6 is easier than Question 5. Do you agree?

$\frac{3}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{3\sqrt{6}}{6}$

$= \frac{\sqrt{6}}{2}$

$\frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}} = 1$

7.(a) $\frac{\sqrt{6}}{2}$
(d) $2\sqrt{21}$
(g) $\frac{\sqrt{30}}{2}$

$$\frac{3}{\sqrt{5}-\sqrt{2}}$$

$$= \frac{3}{\sqrt{5}-\sqrt{2}} \times \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}$$

$$= \frac{3(\sqrt{5}+\sqrt{2})}{5-2}$$

$$= \frac{3(\sqrt{5}+\sqrt{2})}{3}$$

$$= \sqrt{5} + \sqrt{2}$$

10. Rationalize the denominator and evaluate.

(a) $\dfrac{3}{\sqrt{5}-\sqrt{2}}$

(b) $\dfrac{35}{2\sqrt{3}+\sqrt{5}}$

(c) $\dfrac{50}{2\sqrt{7}-\sqrt{3}}$

(d) $\dfrac{14}{3+\sqrt{2}}$

(e) $\dfrac{\sqrt{3}}{\sqrt{7}-2}$

(f) $\dfrac{\sqrt{5}}{\sqrt{11}-\sqrt{3}}$

(g) $\dfrac{21\sqrt{3}}{2\sqrt{5}+\sqrt{13}}$

(h) $\dfrac{-5\sqrt{2}}{\sqrt{14}-3}$

(i) $\dfrac{15\sqrt{3}}{4+\sqrt{7}}$

(j) $\dfrac{-7\sqrt{2}}{\sqrt{13}-\sqrt{6}}$

11. Rationalize the denominator and simplify.

(a) $\dfrac{4}{\sqrt{5}}$

(b) $\dfrac{2\sqrt{3}}{\sqrt{10}}$

(c) $\dfrac{4}{\sqrt{3}-\sqrt{5}}$

(d) $\dfrac{5}{2\sqrt{3}+\sqrt{2}}$

(e) $\dfrac{6}{\sqrt{14}}$

(f) $\dfrac{4\sqrt{3}}{2\sqrt{3}-\sqrt{5}}$

(g) $\dfrac{8}{2\sqrt{5}-3\sqrt{2}}$

(h) $\dfrac{3}{2\sqrt{7}-\sqrt{7}}$

(i) $\dfrac{7}{3\sqrt{5}-\sqrt{5}}$

(j) $\dfrac{\sqrt{5}}{12\sqrt{3}-4\sqrt{7}}$

Were you fooled in 11(h) and 11(i)?

10.(a) $\sqrt{5} + \sqrt{2} = 3.65$
(c) $2(2\sqrt{7} + \sqrt{3}) = 14.048$

11.(a) $\frac{4\sqrt{5}}{5}$
(c) $-2(\sqrt{3} + \sqrt{5})$
(g) $4(2\sqrt{5} + 3\sqrt{2})$
(i) $\frac{7}{10}\sqrt{5}$

Maintaining Basic Skills 1

1. Simplify.
   (a) $a^6 \times a^4$
   (b) $2b^3 \times 5b^2$
   (c) $(6y)(4y^3)(y^4)$
   (d) $-3r \times 4r^4$
   (e) $(4s)(-2s^3)(-3s^4)$
   (f) $5^3 \times 5^7 \times 5^4$

2. Simplify.
   (a) $(c^{-7})(c^2)$
   (b) $f^{\frac{1}{4}} \times f^{\frac{3}{4}}$
   (c) $(g^4)(2g^3)(3g^{-5})$
   (d) $y^{\frac{6}{5}} \times y^{-\frac{4}{5}}$
   (e) $(p^5)(p^{-3})(p^{\frac{1}{2}})$
   (f) $(5a^4)^3$

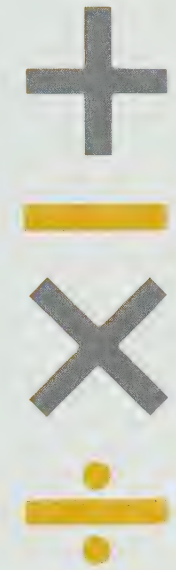
3. Simplify.
   (a) $\dfrac{b^9}{b^4}$
   (b) $\dfrac{8d^6}{4d^3}$
   (c) $\dfrac{-18a^2b^7}{3ab^3}$
   (d) $\dfrac{(p^4)(p^5)}{p^3}$
   (e) $\dfrac{12y^8}{-4y^{-2}}$
   (f) $\dfrac{15t^{-2} \times 4t^8}{10t^4}$

4. Simplify.
   (a) $(w^{-2})^3(w^{\frac{3}{4}})^8$
   (b) $\left(\dfrac{-4d}{e}\right)^2 \div \left(\dfrac{-4d}{e}\right)^6$
   (c) $\left(\dfrac{2}{5}\right)^7 \div \left(\dfrac{2}{5}\right)^5$
   (d) $\left(\dfrac{a}{b}\right)^8 \div \left(\dfrac{a}{b}\right)^{-3}$

5. Simplify.
   (a) $4.0 \times 10^6 \times 2.0 \times 10^8$
   (b) $3.0 \times 10^{-3} \times 3.4 \times 10^5$
   (c) $\dfrac{6.0 \times 10^5 \times 10^{-7} \times 2.0 \times 10^3}{3.0 \times 10^4}$
   (d) $\dfrac{1.2 \times 10^9 \times 2.1 \times 10^5}{3.0 \times 10^4}$

6. Evaluate.
   (a) $16^{\frac{1}{2}}$
   (b) 4^0
   (c) 9^{-2}
   (d) $27^{\frac{2}{3}}$
   (e) $\dfrac{6^9}{6^9}$
   (f) $8^{-\frac{2}{3}}$
   (g) $(-32)^{\frac{1}{5}}$
   (h) $\left(1\dfrac{9}{16}\right)^{\frac{1}{2}}$
   (i) $\left(\dfrac{4}{25}\right)^{0.5}$

7. Simplify.
   (a) $(16x^4)^{\frac{1}{2}}$
   (b) $(125a^6b^3c^9)^{\frac{1}{3}}$
   (c) $(64y^4)^{-\frac{1}{2}}$
   (d) $(8m^3)^{\frac{2}{3}}$
   (e) $(1000n^6)^{-\frac{1}{3}}$
   (f) $\left(\dfrac{36p^8q^{-4}}{121r^6}\right)^{\frac{1}{2}}$



1.(a) 10^{10} (b) $10b^5$ (c) $24y^8$
2.(d) $y^{\frac{2}{5}}$ (e) $p^{-\frac{5}{2}}$ (f) $125a^{12}$
3.(a) b^5 (b) $2d^3$
4.(a) 1 (c) $\frac{4}{25}$
5.(a) 8.0×10^{14} (b) 10.2×10^2
6.(a) 4 (d) 9 (g) -2
7.(a) $4x^2$ (c) $\frac{1}{8y^3}$ (e) $\frac{1}{10n^2}$

8. Change to standard form and then estimate your answer.

   (*a*) $82\,000 \times 1320$ (*d*) $\dfrac{835}{0.051}$

   (*b*) $0.000\,35 \times 2800$ (*e*) $\dfrac{0.0038}{0.022}$

   (*c*) 0.0027×0.0031

9. Convert to standard form and use either slide rule or tables to calculate the answer.

   (*a*) 727×325 (*c*) 0.063×0.472 (*e*) $\dfrac{5370}{0.074}$

   (*b*) $0.006\,23 \times 415$ (*d*) $\dfrac{832\,000}{3.94}$ (*f*) $\dfrac{42.6}{1420}$

10. Simplify.

   (*a*) $5y - 2y^2 - 7y^2 - 3y + 8y^2$

   (*b*) $7g^2h - 3gh^2 - 5g^2h + 9gh^2$

   (*c*) $5y - 3(2 - 3y) - (4y + 3)$

   (*d*) $5(p^2 - 6) + 3(2p^2 - 4) - (8 - p^2)$

11. Expand.

   (*a*) $(p - 3)(p + 5)$ (*d*) $(2r + 7)(3r - 2)$

   (*b*) $(2x - 5)(x - 7)$ (*e*) $(w + 8)(w - 8)$

   (*c*) $(x - 8)^2$ (*f*) $(7x - 2)(7x - 3)$

12. Expand.

   (*a*) $(q^2 + 3qr + 9r^2)(q - 3r)$

   (*b*) $(x + 2)(x^3 + 2x^2 - 3x + 1)$

   (*c*) $(d + f)(d^2 + 2df + f^2)$

   (*d*) $(x - 6)(x + 4)(x - 5)$

13. Divide.

   (*a*) $\dfrac{6c^3 - 4c}{-2c}$ (*d*) $\dfrac{5pq + 10p^2q^2}{5pq}$

   (*b*) $\dfrac{k^2 + 2k}{-k}$ (*e*) $\dfrac{4b^3c - 6b^2c^2 + 2b^3c^2}{2b^2c}$

   (*c*) $\dfrac{6a^3 + 12a^2}{3a^2}$ (*f*) $\dfrac{4d^3 + 8d^2 - 4d}{-4d}$

14. Divide by long division.

   (*a*) $\dfrac{x^3 + 2x^2 - 9x - 18}{x + 2}$ (*c*) $\dfrac{x^3 - 6x^2 + 5x + 6}{x - 2}$

   (*b*) $\dfrac{x^3 - x^2 - 6x}{x - 3}$ (*d*) $\dfrac{2x^3 - 5x^2 + 18}{2x + 3}$

8.(a) 8×10^7 (d) 1.7×10^4
9.(b) 2.585 (e) 72 560
10.(a) $2y - y^2$ (c) $10y - 9$
11.(a) $p^2 + 2p - 15$
(d) $6r^2 + 17r - 14$
12.(a) $q^3 - 27r^3$
(b) $x^4 + 4x^3 + x^2 - 5x + 2$
13.(a) $-3c^2 + 2$ (d) $1 + 2pq$
14.(a) $x^2 - 9$ (b) $x^2 + 2x$

15. Expand and simplify.
   (*a*) $(x-6)(x-5)-(x+4)(x+3)$
   (*b*) $(2y-1)(y+2)-(y-3)(y-4)$
   (*c*) $(2a-3b)(2a+3b)-(a+b)(a-b)$
   (*d*) $(4y+3)(y-7)+(2y-5)(2y-1)$
   (*e*) $4p-3p(p+2)+(p+2)(p-4)$
   (*f*) $r(r^2-5)-2r(r+3)(r-3)$

16. Factor.
   (*a*) $6p^2-2p$
   (*b*) $5m+10$
   (*c*) $3k^2-6k$
   (*d*) $(a+2b)k+1(a+2b)$
   (*e*) $(x+y)(a+b)+3(x+y)$
   (*f*) $4k^2-k(y+z)$

17. Factor.
   (*a*) a^2-9
   (*b*) b^2-25
   (*c*) $4c^2-16$
   (*d*) $1-100w^2$
   (*e*) $(a+b)^2-c^2$
   (*f*) $g^2-(h-k)^2$

18. Factor.
   (*a*) $x^2-8x+15$
   (*b*) y^2-6y+9
   (*c*) t^2-5t-6
   (*d*) $1-4x-60x^2$
   (*e*) $a^2-4ab+4b^2$
   (*f*) $7+8k+k^2$

19. Factor.
   (*a*) $rp+rs+pt+ts$
   (*b*) $3x+6y-kx-2ky$
   (*c*) $(a+b)^2+3a+3b$
   (*d*) $s^2-t^2-4s+4t$
   (*e*) $x^3-5x^2+3x-15$
   (*f*) y^3+4y^2-y-4

20. Factor completely.
   (*a*) $3p^2-12$
   (*b*) $3m^2-15m+18$
   (*c*) x^4-1
   (*d*) $3p^2-3p-18$
   (*e*) $40a^3-5$
   (*f*) $3x^2-24x+48$

21. Simplify.
   (*a*) $\frac{2}{3}\times\frac{3}{4}$
   (*b*) $\frac{5}{6}\times\frac{1}{-2}$
   (*c*) $\frac{5pr}{4q}\div\frac{2q}{5r}$
   (*d*) $\frac{5abc}{7a^2}\times\frac{-21a}{10c}$
   (*e*) $\frac{x^4y^3}{-xz}\div\frac{-x^2y}{z^3}$
   (*f*) $\frac{8a^3b^2}{2a^5c}\times\frac{9a^2c^3}{6b^6}$
   (*g*) $\frac{(x+9)(x+3)}{(x+8)(x+9)}\times\frac{x(x+8)}{(x+3)(x+7)}$
   (*h*) $\frac{(a+2b)(2a+3b)}{(2a+3b)}\div\frac{(a+2b)(a+5b)}{(a+5b)(6a+7b)}$

15.(a) $-18x+18$
(e) $-2p^2-4p-8$
(f) $-r^2+13r$

16.(a) $2p(3p-1)$
(d) $(a+2b)(k+1)$

17.(a) $(a+3)(a-3)$
(b) $(b+5)(b-5)$
(f) $(g+h-k)(g-h+k)$

18.(a) $(x-5)(x-3)$
(b) $(y-3)^2$
(c) $(t+1)(t-6)$

19.(a) $(r+t)(p+s)$
(b) $(x+2y)(3-k)$
(c) $(a+b)(a+b+3)$

20.(d) $3(p+2)(p-3)$
(e) $5(2a-1)(4a^2+2a+1)$
(f) $3(x-4)^2$

21.(b) $-\frac{5}{3}$ (f) $\frac{6c^2}{b^1}$
(d) $-\frac{3b}{2}$ (h) $6a+7b$

22. Simplify.

(a) $\dfrac{6y^2 - 10y}{2y}$ (b) $\dfrac{x^2 - 5x + 6}{x^2 - 3x}$ (c) $\dfrac{y^3 - 2y^2 + y}{y^2 - y}$

23. Simplify.

(a) $\dfrac{a^2 - 9}{a^2 + 6a + 9} \times \dfrac{a^2 + 3a}{a}$ (b) $\dfrac{g^2 - 2g - 3}{g^2 - 4g - 5} \times \dfrac{g^2 - 9g + 20}{g^2 - 7g + 12}$

(c) $\dfrac{2a^2 + 7ab + 3b^2}{6a^2 + 2ab} \div \dfrac{2a^2 + 5ab - 3b^2}{9a^2 + 3ab}$

(d) $\dfrac{4x^2 + 20x + 25}{4x^2 - 25} \div \dfrac{2x^2 + 7x + 5}{2x^2 - 3x - 5}$

24. Simplify.

(a) $\dfrac{2}{3} + \dfrac{7}{2} - \dfrac{5}{6}$ (c) $\dfrac{1}{2a} + \dfrac{1}{ab}$

(b) $1\dfrac{3}{4} - 2\dfrac{2}{3} - 3\dfrac{1}{2}$ (d) $\dfrac{3}{c^2 d} - \dfrac{4}{cd^2}$

25. Simplify.

(a) $\dfrac{x + 2}{3} + \dfrac{x + 5}{4}$ (d) $\dfrac{3x}{x - 2y} - 3$

(b) $\dfrac{y + 1}{6} - \dfrac{y + 2}{9}$ (e) $\dfrac{x + 2}{x - 3} - \dfrac{x - 1}{x - 4}$

(c) $\dfrac{a + b}{15} - \dfrac{a - b}{10} + \dfrac{7a}{30}$ (f) $\dfrac{3}{a^2 - ab} - \dfrac{2}{a^2 - b^2}$

26. Solve.

(a) $(x - 7)(2x + 3) = 0$ (d) $y^2 - 8y + 15 = 0$

(b) $x^2 - 7x + 10 = 0$ (e) $3x^2 - 5x - 2 = 0$

(c) $x^2 - 7x = 0$ (f) $25w^2 - 16 = 0$

27. Multiply.

(a) $\sqrt{7} \times \sqrt{11}$ (d) $3\sqrt{2} \times 4\sqrt{3} \times \sqrt{5}$

(b) $4\sqrt{5} \times \sqrt{5}$ (e) $10\sqrt{7} \times 2\sqrt{3} \times 5\sqrt{2}$

(c) $(2\sqrt{3})^2$ (f) $p\sqrt{q} \times r\sqrt{q}$

28. Simplify.

(a) $\dfrac{\sqrt{14}}{\sqrt{7}}$ (c) $\dfrac{14\sqrt{15}}{12\sqrt{35}}$ (e) $\sqrt{\dfrac{xy^2}{z}} \times \sqrt{\dfrac{z}{xy}}$

(b) $\dfrac{3\sqrt{30}}{\sqrt{6}}$ (d) $\sqrt{\dfrac{3}{13}} \times \sqrt{\dfrac{26}{15}}$ (f) $8\sqrt{\dfrac{b}{c}} \div \left(4\sqrt{\dfrac{e}{f}}\right)$

22.(a) $3y - 5$

23.(a) $a - 3$
(b) 1

24.(a) $\frac{10}{3}$ (b) $-\frac{53}{12}$

25.(a) $\frac{7x + 23}{12}$ (d) $\frac{6y}{x - 2y}$

26.(a) $7, -\frac{3}{2}$ (d) 5, 3

27.(b) 20 (f) prq
(c) 12

28.(a) $\sqrt{2}$ (e) $\sqrt{y}$
(b) $3\sqrt{5}$

29. Express in the simplest form.

(*a*) $\sqrt{12}$ (*c*) $5\sqrt{63}$ (*e*) $\sqrt{\frac{9}{4}}$

(*b*) $\sqrt{45}$ (*d*) $\sqrt{x^4y^5}$ (*f*) $5\sqrt{\frac{121x^2}{16}}$

30. Add.

(*a*) $5\sqrt{13} - 3\sqrt{13} + \sqrt{13} + 4\sqrt{13}$

(*b*) $2\sqrt{7} + 3\sqrt{5} + 4\sqrt{7} - \sqrt{5}$

(*c*) $6\sqrt{27} + 5\sqrt{12} - \sqrt{48} + 2\sqrt{75}$

(*d*) $2\sqrt{125} + 4\sqrt{63} + 3\sqrt{20} - \sqrt{28}$

31. Expand.

(*a*) $\sqrt{5}(\sqrt{2} + \sqrt{3})$ (*d*) $\sqrt{11}(2\sqrt{3} + 5)$

(*b*) $\sqrt{6}(4 + \sqrt{5})$ (*e*) $4\sqrt{2}(3\sqrt{5} - 7)$

(*c*) $\sqrt{7}(\sqrt{2} + 1)$ (*f*) $5\sqrt{3}(2\sqrt{5} + 3\sqrt{7})$

32. Expand.

(*a*) $(\sqrt{2} + \sqrt{3})(\sqrt{5} + \sqrt{6})$ (*d*) $(2\sqrt{3} + 4)(\sqrt{3} - \sqrt{5})$

(*b*) $(\sqrt{7} - \sqrt{2})(\sqrt{7} + \sqrt{2})$ (*e*) $(4\sqrt{2} + 5)(3\sqrt{2} + 2)$

(*c*) $(\sqrt{5} + \sqrt{6})^2$ (*f*) $(\sqrt{6} + \sqrt{2})(\sqrt{7} + \sqrt{5})$

33. Expand and simplify.

(*a*) $(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})$ (*c*) $(\sqrt{10} + \sqrt{6})(\sqrt{2} + \sqrt{5})$

(*b*) $(\sqrt{14} + 4)(\sqrt{21} - 5)$ (*d*) $(2\sqrt{10} - 4\sqrt{15})(3\sqrt{5} + 5\sqrt{6})$

34. Rationalize the denominator.

(*a*) $\frac{\sqrt{7}}{\sqrt{11}}$ (*c*) $\frac{7\sqrt{5}}{4\sqrt{2}}$ (*e*) $\frac{10\sqrt{7}}{3\sqrt{5}}$

(*b*) $\frac{8\sqrt{6}}{\sqrt{7}}$ (*d*) $\frac{2\sqrt{13}}{\sqrt{2}}$ (*f*) $\frac{15\sqrt{17}}{7\sqrt{10}}$

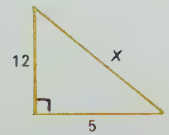
35. Rationalize the denominator.

(*a*) $\frac{1}{\sqrt{7} - \sqrt{2}}$ (*c*) $\frac{4}{4 + \sqrt{6}}$ (*e*) $\frac{26}{2\sqrt{3} + 5}$

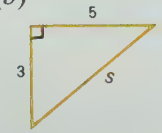
(*b*) $\frac{1}{\sqrt{13} + 2}$ (*d*) $\frac{3\sqrt{5}}{\sqrt{6} - \sqrt{3}}$ (*f*) $\frac{4\sqrt{3}}{4\sqrt{5} - 3\sqrt{2}}$

36. Find the value of each variable, leaving irrational numbers in radical form.

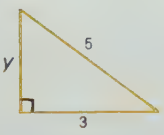
(*a*)



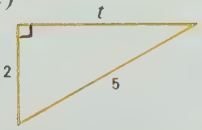
(*b*)



(*c*)



(*d*)



29. (a) $2\sqrt{3}$ (d) $x^2y^2\sqrt{y}$

30. (a) $7\sqrt{13}$
(b) $6\sqrt{7} + 2\sqrt{5}$

31. (a) $\sqrt{10} + \sqrt{15}$
(d) $2\sqrt{33} + 5\sqrt{11}$

32. (b) 5
(c) $11 + 2\sqrt{30}$

33. (a) 4

34. (a) $\frac{\sqrt{77}}{11}$ (c) $\frac{7\sqrt{10}}{8}$
(b) $\frac{8\sqrt{42}}{7}$

35. (a) $\frac{\sqrt{7} + \sqrt{2}}{5}$
(d) $\sqrt{30} + \sqrt{15}$

36. (a) 13
(b) $\sqrt{34}$

Unit 6
Working With Equations and Inequations – One Variable

6.1 Review of Equations in One Variable

A *Adding Terms to Isolate the Variable*

Solve $x + 7 = 10.$

Solution

$$x + 7 - 7 = 10 - 7$$
$$\therefore x = 3$$
$$\therefore \text{solution is } x = 3$$

Add -7 to both sides.

B *Multiplying Each Side by the Same Number*

Solve $3x - 5 = 10.$

Solution

$$3x - 5 = 10$$
$$3x - 5 + 5 = 10 + 5$$
$$\therefore 3x = 15$$
$$\frac{1}{3_1}3x \times \frac{1}{3_1} = \frac{5}{15} \times \frac{1}{3_1}$$
$$\therefore x = 5$$
$$\therefore \text{solution is } x = 5$$

Multiply by $\frac{1}{3}$ to solve for x.

$2x + 3 = 9$
Subtract 3 from each side
$2x = 6$
Multiply each side by $\frac{1}{2}$.
$x = 3$

1. Solve.

(a) $x + 2 = 7$
(b) $3 + y = 6$
(c) $z - 4 = 2$
(d) $a + 3 = -1$
(e) $2x + 3 = 9$
(f) $3y - 7 = 8$
(g) $5z + 6 = 1$
(h) $4 - 2x = 3$
(i) $7x + 4 = 6$
(j) $3 - 9z = 30$

C *Removing Brackets*

Solve $4(x + 1) = 2(x + 5)$.

Solution

$$4x + 4 = 2x + 10$$ ← MULTIPLY TO CLEAR BRACKETS

$$4x + 4 - 4 = 2x + 10 - 4$$

$$4x = 2x + 6$$

$$4x - 2x = 2x - 2x + 6$$ ← SUBTRACT 4 AND THEN $2x$ FROM BOTH SIDES

$$2x = 6$$

$$\cancel{2}^{1}x \times \frac{1}{\cancel{2}_{1}} = \cancel{6}^{3} \times \frac{1}{\cancel{2}_{1}}$$ ← MULTIPLY BOTH SIDES BY $\frac{1}{2}$

$$\therefore x = 3$$

$\therefore$ solution is $x = 3$

D *Multiplying to Clear Fractions*

Solve $\frac{x}{3} + 3 = \frac{1}{4}$.

Solution

$$12\left(\frac{x}{3} + 3\right) = {}^{3}12\left(\frac{1}{\cancel{4}_{1}}\right)$$ MULTIPLY BY 12 TO ELIMINATE THE DENOMINATORS 3 AND 4

$$\frac{{}^{4}\cancel{12}x}{\cancel{3}_{1}} + 36 = 3$$

$$4x + 36 = 3$$

$$4x = 3 - 36$$

$$4x = -33$$

$$x = \frac{-33}{4}$$

$\therefore$ solution is $x = \frac{-33}{4}$

1. Solve.

(a) $3(x-3) = x+7$

(b) $5(a+2) = 2(2a+3)$

(c) $4(c-3) = 3(2c+7)$

(d) $7(3-2d) = 5(5-3d)$

(e) $\frac{x}{7} + 3 = 4$

(f) $\frac{5a}{3} + \frac{2}{5} = 2$

(g) $\frac{2x}{7} - \frac{3}{5} = \frac{29}{35}$

(h) $\frac{1}{3}(2x-5) = \frac{1}{5}(x+8)$

(i) $\frac{2}{7}(3y+1) = \frac{1}{2}(y+2)$

(j) $\frac{2}{3}(z-1) = \frac{5}{6}(z+2)$

Strategy

(a) Remove any brackets and collect like terms.
(b) Multiply to clear fractions.
(c) Add terms to each side to isolate the variable.
(d) If necessary multiply each side by the same number to solve for the variable.

Exercise 6.1

1. Solve.

(a) $x + 7 = 14$
(b) $y - 2 = 7$
(c) $5x + 3 = 13$
(d) $2m + 7 = 9$
(e) $3e - 2 = 7$
(f) $4c + 9 = -3$
(g) $2f + 7 = -23$
(h) $17r + 4 = -13$
(i) $9x - 7 = 4$
(j) $2t + 11 = 42$

2. Solve.

(a) $5(3x+7) + 9 = 59$
(b) $7(5x-4) = 2(x+19)$
(c) $3(2m+1) - 2(m+7) = 7(m+3) + 1$
(d) $5(2a-7) + 3(4a-3) = 2a + 7(2a-2)$
(e) $2y - 3(5y+7) = 5(4-3y) + 3$
(f) $3(4-2c) + 7c = 5(3c-6)$
(g) $7(x+2) - 2(x-7) = 3(x+12)$
(h) $9(2b-3) + 7(3b+2) = 5(6b-2)$
(i) $4(2e+7) - 3(5e+3) = -2$
(j) $7(5g+7) - 4(g+7) = -10$

1.(a) 7 (b) 9 (f) −3 (g) −15
2.(a) 1 (d) 5 (e) 22 (i) 3 (j) −1

3. Solve.

(a) $\frac{x+3}{7} = \frac{23}{7}$

(b) $\frac{3e-7}{3} = \frac{2e-5}{4}$

(c) $\frac{x}{5} + \frac{3}{4} = \frac{5x}{2} - \frac{4}{10}$

(d) $\frac{3}{7}(c-3) = \frac{1}{14}(3c+6)$

(e) $\frac{2}{3}(3y-5) + 2 = \frac{1}{2}(y-2)$

(f) $\frac{3}{5}(2e+7) + \frac{1}{3} = e + \frac{1}{5}(2e+20)$

(g) $\frac{7}{11}(3c-4) + 3 = 2c + \frac{1}{2}(c+2)$

(h) $\frac{5a+2}{5} = \frac{2a+5}{3} - 1$

(i) $\frac{4(3x+1)}{3} = \frac{2(5x-7)}{2} + \frac{1}{3}$

(j) $\frac{(5-2e)}{2} = \frac{3(e+2)}{7} + \frac{3}{14}$

4. Solve for x.

(a) $ax + a = 3a$

(b) $2bx + b = 4b$

(c) $(a+b)x = a + b$

(d) $(2c+d)x = 2c + d$

(e) $2ax + 2b = bx + 4a$

(f) $5x - ax = 15 - 3a$

(g) $ax + 2ax = 7 - 4ax$

(h) $9bx - 3bx = 5bx + 1$

(i) $2ax + 5x = 2a + 5$

(j) $3cx + 16 = 4x + 12c$

3.(a) 20 (i) 8
(d) 8 (j) 1

4.(c) 1 (f) 3
(d) 1 (i) 1

$a x + a - a = 3a - a$

$ax = 2a$

$\frac{1}{a} x \times \frac{1}{a-1} = 2 \frac{1}{a} \times \frac{1}{a-1}$

$x = 2$

6.2 Solving Inequations in One Variable

Inequations are solved in the same manner as equations with one exception. When multiplying or dividing by a negative, you must change the direction of the inequality sign. This can be demonstrated using the inequation $7 > 3$.

(*a*) You can add.

$7 > 3$ is true.

$7 + 3 > 3 + 3$

(*or*) $10 > 6$ is still true.

(*b*) You can subtract.

$7 - 4 > 3 - 4$

(*or*) $3 > -1$ is true.

(*c*) You can multiply.

$7 \times 3 > 3 \times 3$

(*or*) $21 > 9$ is true.

(*d*) You can divide.

$7 \div 7 > 3 \div 7$

(*or*) $1 > \frac{3}{7}$ is true.

(*But*)

(*e*) You cannot multiply by a negative.

$7 \times (-2) > 3 \times (-2)$

since $-14 > -6$ is false.

(*f*) You cannot divide by a negative.

$7 \div (-7) > 3 \div (-7)$

since $-1 > \frac{-3}{7}$ is false.

You must

Change the direction of the sign.

(*e*) $-14 < -6$ is true.

(*f*) $-1 < \frac{-3}{7}$ is true.

When multiplying or dividing by a negative, change

$>$ to $<$
and
$<$ to $>$

$\in$ means "is an element of"

$N = \{1, 2, 3, \ldots\}$

Q means "the set of rationals"

Example 1

Solve the inequation $x + 5 > 9$, where $x \in N$.

Solution

$$\begin{aligned} x + 5 &> 9 \\ \therefore\ x &> 9 - 5 \qquad \text{Subtract 5.} \\ \therefore\ x &> 4 \end{aligned}$$

$\therefore$ the solution set is $\{5, 6, 7, \ldots\}$.

Example 2

Solve the inequation $4x - 6 < 2x + 1$, where $x \in Q$.

Solution

$$\begin{aligned} 4x - 6 &< 2x + 1 \\ 4x - 2x &< 1 + 6 \qquad \text{Subtract } 2x \text{ and} \\ 2x &< 7 \qquad\qquad\ \ \text{add 6.} \\ \therefore\ x &< \frac{7}{2} \qquad\quad \text{Divide by 2.} \end{aligned}$$

$\therefore$ the solution set is $\{x \mid x < \frac{7}{2}, x \in Q\}$.

THE SET OF ALL x — SUCH THAT

Example 3

Solve the inequation $3 - 3x > x + 5$, where $x \in Q$.

Solution

$$\begin{aligned} 3 - 3x &> x + 5 \\ -3x - x &> 5 - 3 \qquad \text{Subtract } x \text{ and } 3. \\ -4x &> 2 \\ *x &< \frac{-1}{2} \qquad \text{Divide by } -4. \end{aligned}$$

$\therefore$ the solution set is $\{x \mid x < \frac{-1}{2}, x \in Q\}$.

* Notice the change in sign!

$I = \{\ldots, -2, -1, 0, 1 \ldots\}$

Example 4

Solve the inequation $-x - 5 > 2 - \frac{2}{3}x$, $x \in I$.

Solution

$$-x - 5 > 2 - \frac{2}{3}x \qquad \text{Add } \frac{2}{3}x \text{ and } 5.$$

$$-x + \frac{2}{3}x > 2 + 5$$

$$-\frac{1}{3}x > 7$$

$$x < -21 \qquad \text{Multiply by } -3.$$

$\therefore$ the solution set is $\{\ldots, -24, -23, -22\}$.

Notice the change in sign!

Exercise 6.2

Discussion

1. Solve.
   (*a*) $x + 3 > 7$
   (*b*) $y + 5 < 8$
   (*c*) $7 + z > 3$
   (*d*) $6 + 2x < 4$
   (*e*) $5 + 3y > 3$
   (*f*) $1 + 7c \leqslant 6$

2. Solve.
   (*a*) $5 - x < 2$
   (*b*) $4 + y > 1$
   (*c*) $7 - 3a < 4$
   (*d*) $15 - 3m \leqslant 3$
   (*e*) $4 - 2z \geqslant 8$
   (*f*) $3 + 5x \leqslant -2$

Written Solutions

3. Solve.
   (*a*) $5x + 3 \leqslant 2x + 6$
   (*b*) $8a + 7 \geqslant a - 14$
   (*c*) $3m \leqslant 5m + 6$
   (*d*) $2b + 7 \geqslant 3b + 1$
   (*e*) $14 - 3y > 6 - 5y$
   (*f*) $18 + 4z < 3 - 3z$
   (*g*) $7x - 3 \geqslant 8x + 15$
   (*h*) $14 < 2x + 7$
   (*i*) $4p - 7 < -9p + 1$
   (*j*) $6c + 13 \geqslant 9c + 31$

4. Solve.
   (*a*) $4(x + 2) \geqslant x - 1$
   (*b*) $7(3a - 1) \leqslant 5(4a + 2)$
   (*c*) $3(b + 7) \geqslant 4b + 1$
   (*d*) $2(x + 6) + 3x \leqslant 7(x + 3)$
   (*e*) $4(3y - 2) \geqslant 5(3y + 1) + 2$
   (*f*) $5(m - 3) + 6 \leqslant m + 3(m - 2)$
   (*g*) $c + 3(c - 2) > 5(c + 7)$

3.(a) $x \leqslant 1$ (g) $x \leqslant -18$
(c) $m \geqslant -3$ (j) $c \leqslant -6$
(e) $y < -4$

4.(a) $x \geqslant -3$ (h) $x > -8$
(b) $a \leqslant 17$ (i) $a < 30$

(h) $\frac{1}{2}(x-2) < \frac{1}{3}(2x+1)$

(i) $\frac{1}{5}(2a-5) < \frac{1}{3}(a+3)$

5. Solve.

(a) $\frac{x}{3}+\frac{1}{2} < \frac{2}{3}$

(b) $\frac{y}{5}-\frac{3}{2} > 5$

(c) $\frac{5}{7}-\frac{2a}{3} < \frac{1}{3}$

(d) $\frac{14}{3}-\frac{a}{4} \geqslant \frac{3a}{6}+\frac{1}{6}$

(e) $\frac{23}{5}+\frac{3m}{2} \leqslant 2m-\frac{4}{10}$

(f) $\frac{7e}{3}+\frac{9}{2} \leqslant \frac{e}{2}+\frac{5}{6}$

(g) $\frac{2}{15}-\frac{5x}{6} \geqslant \frac{22}{15}+\frac{x}{2}$

(h) $\frac{7}{8}+\frac{5a}{12} \leqslant \frac{7}{3}$

(i) $\frac{6}{7}-\frac{x}{3} > \frac{1}{2}$

(j) $\frac{a}{9}-\frac{5}{6} < \frac{a}{2}-\frac{1}{18}$

5.(d) $a \leqslant 6$ (g) $x \leqslant -1$
(e) $m \geqslant 10$ (j) $a > -2$

6.3 Graphing the Solution Set of Equations and Inequations

The solution sets of all equations and inequations involving one variable may be graphed on a number line.

Example 1

Graph the solution set of

$$\frac{x}{3}+\frac{7}{2}=\frac{5}{3},\ x \in R.$$

Solution

Why 6? →

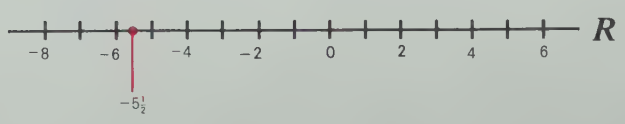
Multiply by 6. $6\left(\frac{x}{3}+\frac{7}{2}\right)=6\left(\frac{5}{3}\right)$

$$\cancel{6}^{2}\left(\frac{x}{\cancel{3}_{1}}\right)+\cancel{6}^{3}\left(\frac{7}{\cancel{2}_{1}}\right)=\cancel{6}^{2}\left(\frac{5}{\cancel{3}_{1}}\right)$$

$$2x+21=10$$

$$2x=-11$$

$$x=\frac{-11}{2} \text{ or } -5\tfrac{1}{2}$$



R means the real number system (rationals plus irrationals).

Example 2

Graph the solution set of

$$\frac{3x}{4} + \frac{7}{3} > \frac{5}{6},\ x \in R.$$

Solution

Multiply by 12. $$12\left(\frac{3x}{4} + \frac{7}{3}\right) > 12\left(\frac{5}{6}\right)$$

$$\cancel{12}^{3}\left(\frac{3x}{\cancel{4}_{1}}\right) + \cancel{12}^{4}\left(\frac{7}{\cancel{3}_{1}}\right) > \cancel{12}^{2}\left(\frac{5}{\cancel{6}_{1}}\right)$$

$$9x + 28 > 10$$
$$9x > -18$$
$$x > -2$$

$\therefore$ the solution set is $\{x \mid x > -2,\ x \in R\}$.

* The arrow shows that the graph goes on forever.

* The open dot means that −2 is NOT included.

Example 3

Graph the solution set of

$$\frac{c}{4} - \frac{7}{6} \geqslant \frac{4c}{3} + 1,\ c \in R$$

Solution

Multiply by 12. $$12\left(\frac{c}{4} - \frac{7}{6}\right) \geqslant 12\left(\frac{4c}{3} + 1\right)$$

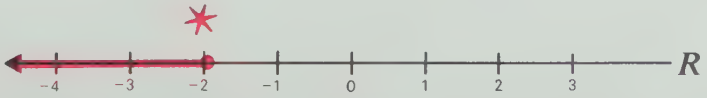
$$12\left(\frac{c}{4}\right) - 12\left(\frac{7}{6}\right) \geqslant 12\left(\frac{4c}{3}\right) + 12(1)$$

$$3c - 14 \geqslant 16c + 12$$
$$3c - 16c \geqslant 12 + 14$$
$$-13c \geqslant 26$$

$$c \leqslant \frac{26}{-13} *$$

$$c \leqslant -2$$

$\therefore$ the solution set is $\{c \mid c \leqslant -2,\quad c \in R\}$.



* Remember multiplying or dividing by a negative?

* The dot is solid because −2 IS included.

Exercise 6.3

Discussion

(Assume all variables are real numbers.)

1. Discuss the graphs.
   (*a*) $x + 2 > 5$
   (*b*) $s - 3 < -2$
   (*c*) $2m \geq 6$
   (*d*) $5c - 7 \leq 3$
   (*e*) $2y + 9 \geq 3 + y$
   (*f*) $4e + 6 \leq e$

2. Discuss the graphs.
   (*a*) $3x - 2 \leq 5x$
   (*b*) $2a + 7 \geq 5$
   (*c*) $5z - 6 < 6z$
   (*d*) $2r > 4r - 6$
   (*e*) $4t \geq 3(2t + 4)$
   (*f*) $3e - 7 < 10e$

Written Solutions

(Assume all variables are real numbers.)

3. Graph the solution set on a real number line.
   (*a*) $3x + 7 \leq -14 - 4x$
   (*b*) $2e - 5 > 5e + 4$
   (*c*) $2(c + 4) \leq 5(c + 1)$
   (*d*) $6(3f + 5) = 2(4f - 5)$
   (*e*) $7(2x + 3) - 7 > 6(4x + 9)$
   (*f*) $(6x + 1) - 3 = 12x - 4(2x - 1)$
   (*g*) $5y - (7 + 3y) + 2y + (2y - 3) < 14$
   (*h*) $7(2z - 3) - 11(3 - 2z) \geq 18$
   (*i*) $m - 5 \leq 8 - 2(m + 2)$
   (*j*) $2(7 + c) \leq 15(c - 1) + 4(c + 3)$

Remember!
If $-17c \leq -17$
then $c \geq 1$.

4. Graph the solution set using a real number line.
   (*a*) $\dfrac{x - 3}{2} < 10$
   (*b*) $\dfrac{a}{3} + \dfrac{a}{4} \geq 7$
   (*c*) $\dfrac{2}{3}z + 1 \leq \dfrac{z}{2}$
   (*d*) $\dfrac{m - 3}{6} \geq \dfrac{m + 1}{7} - \dfrac{2}{3}$
   (*e*) $4 - \dfrac{4n - 3}{2} < \dfrac{n + 20}{3}$

(*f*) $\frac{3}{16}(b-1) - \frac{5}{12}(b-4) \leqslant \frac{2}{3}(b-6) + \frac{5}{48}$

(*g*) $\frac{3+e}{2} > \frac{4e-3}{3}$

(*h*) $\frac{1}{4}(2x-2) - \frac{1}{2}(3x+1) \geqslant \frac{1}{6}(x+1)$

(*i*) $\frac{a}{9} - \frac{5}{6} = -\frac{2}{9} - \frac{a}{2}$

(*j*) $\frac{4}{7}(e+3) \leqslant 2e - \frac{2e-6}{2}$ *

* Careful!

5. Solve the inequation and graph the solution set.

(*a*) $(x+1)(x+2) \leqslant (x-1)^2$

(*b*) $(x+2)(x-3) \geqslant (x-2)(x+2)$

(*c*) $(x+5)(x-1) \leqslant (x+3)^2$

(*d*) $\frac{(a+1)^2}{3} - \frac{(a+2)^2}{2} \leqslant \frac{2-a^2}{6}$

(*e*) $(m-5)^2 + 3(m+2)^2 > 4(m^2-1) - 9$

(*f*) $2b(b+1) - (b-1)^2 \leqslant (b+3)^2$

(*g*) $(2p-5)(p-3) \geqslant p^2 + (p-1)(p+1) - 6$

(*h*) $(y+3)(y-2) < y^2 + 5y + 6$

(*i*) $(3m+1)(m-3) > 2m^2 + (m+1)^2 + 1$

(*j*) $(3x+5)^2 < 5x^2 + (2x-3)^2 + 2$

Use FOIL!

6.4 The "And", "Or" Connectives

AND means "common to."

A *Example 1* "AND"

Given the set $A = \{1, 2, 3, 4, 5, 6\}$ and the set $B = \{4, 5, 6, 7, 8, 9\}$ Which elements are in set A AND set B?

Solution

Since only 4, 5 and 6 are common to both sets, the solution set is $\{4, 5, 6\}$. This is symbolized $A \cap B = \{4, 5, 6\}$ and is often called the "intersection set."

$\cap$ means "intersection" and corresponds to AND

Graphical Solution

$\{x|\ \}$

"the set of all x such that..."

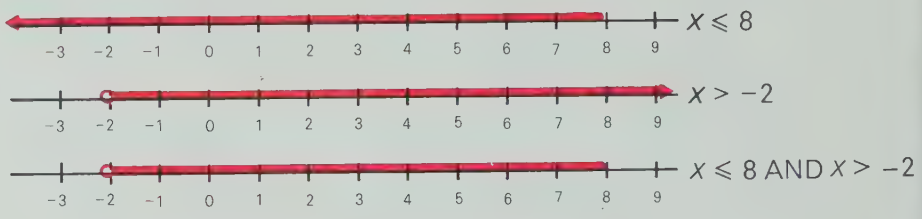
OR means "either".

$\cup$ is the symbol for OR.

Example 2

Find graphically, the real numbers such that $x \leqslant 8$ AND $x > -2$.

Solution



$\therefore$ the solution set may be described as $\{x|-2 < x \leqslant 8\}$.

B *Example 1* "OR"

A physical education teacher believed that all short boys were good wrestlers and all tall boys were good basketball players. In order to select boys for these two activities, he selected boys that were below 167 cm OR above 178 cm. From the following group, which would he select?

{167, 170, 184, 169, 168, 158, 192, 178, 171, 172, 165, 166, 180, 192, 161, 190, 163}.

Solution

The solution set contains elements less than 167 cm OR greater than 178 cm but not necessarily both.

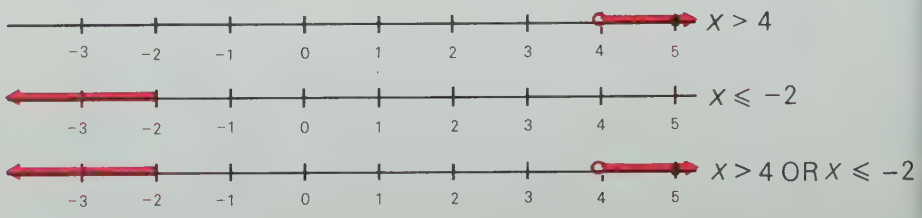
$\therefore$ S.S. is {184, 158, 192, 165, 166, 180, 192, 161, 190, 163}.

Why were 167, 170, 169, 168, 178, 171, 172 not chosen?

Example 2

Find graphically, the real numbers such that $x > 4$ OR $x \leqslant -2$.

Solution



This is the "union set" - it contains everything in each of the given sets.

Exercise 6.4

Discussion

1. Describe in your own words the meaning of the connective AND in mathematics.
2. Describe in your own words the meaning of the connective OR in mathematics.
3. Given $A = \{-3, -2, -1, 0\}$, $B = \{-1, 0, 1, 2\}$, $C = \{1, 2, 3, 4, 5\}$ state the solution set for the following.
   (a) $A \cap B$ (d) $B \cup C$
   (b) $A \cup B$ (e) $A \cap C$
   (c) $B \cap C$ (f) $A \cup C$

$\cap$ (elements must be in both sets)

$\cup$ (includes all elements from each set)

Written Solutions

4. Given $A = \{5, 3, 1, 0\}$, $B = \{1, 5, -5, -1\}$, graph each set and then graph $A \cap B$. Using brace brackets, describe $A \cap B$.
5. Repeat Question 4 for $A \cup B$.
6. If $M = \{7, 4, -3, 2, 9, 11\}$ and $N = \{1, 2, 3, 4, 5\}$ describe the solution set for
   (a) $M \cap N$ (b) $M \cup N$
7. A set $X = \{3, 11, 14, -7, -4, 6\}$ and a second set $Y = \{1, 4, -14, -11, 8\}$. Describe the solution set for
   (a) $X \cap Y$ (b) $X \cup Y$
8. If set $C = \{7, 8, 9, 10\}$ and $C \cap D = \{7, 8\}$ and $C \cup D = \{5, 6, 7, 8, 9, 10\}$, what elements are in set D?
9. Graph the solution set for each of the following. State the solution set using brace brackets.
   (a) $x > 2$ AND $x \leqslant 6$ (i) $x > 5$ AND $x < -3$
   (b) $x \geqslant -3$ AND $x < 0$ (j) $x < 0$ OR $x \geqslant 2$
   (c) $x \geqslant -2$ AND $x > 2$ (k) $x < -2$ OR $x \leqslant -5$
   (d) $x > 5$ OR $x < 3$ (l) $x > 7$ AND $x \geqslant -1$
   (e) $x \geqslant -1$ OR $x < -5$ (m) $x \leqslant 3$ OR $x < -2$
   (f) $x < 10$ AND $x > -1$ (n) $x > 2$ AND $x \leqslant -8$
   (g) $x > 5$ OR $x \geqslant -1$ (o) $x > 5$ AND $x < -7$
   (h) $x < 8$ AND $x \leqslant 1$ (p) $x > 5$ OR $x < -7$

$\emptyset$ means "the empty set"

$\emptyset = \{\ \}$

4. $A = \{5, 3, 1, 0\}$
   $B = \{1, 5, -5, -1\}$
   $A \cap B = \{1, 5\}$
5. $A \cup B = \{-5, -1, 0, 1, 3, 5\}$
6. (a) $\{2, 4\}$
7. (a) ϕ
9. (a) $\{x | 2 < x \leqslant 6\}$
   (c) $\{x | x > 2\}$
   (d) $\{x | x < 3 \text{ or } x > 5\}$
   (g) $\{x | x > -1\}$
   (h) $\{x | x \leqslant 1\}$

6.5 Uses of Factoring

Example 1

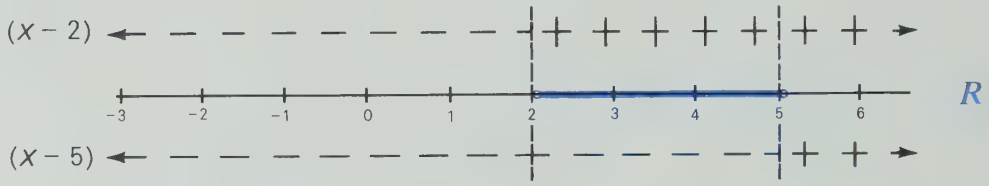
Solve the quadratic inequation $x^2 - 7x + 10 < 0$.

Solution

$$x^2 - 7x + 10 < 0$$
$$\therefore (x - 2)(x - 5) < 0$$

(*a*) Here we have two factors $(x - 2)$ and $(x - 5)$ that have a product that is negative.

(*b*) To have a negative product we must have values of x that give values to the two factors that have opposite signs.



$\therefore$ S.S. is $\{x | 2 < x < 5, x \in R\}$.

Example 2

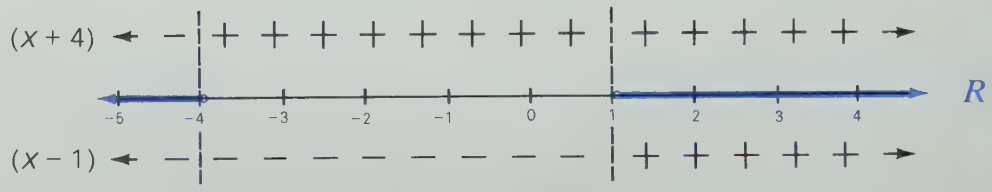
Solve the quadratic inequation $x^2 + 3x - 4 > 0$.

Solution

$$x^2 + 3x - 4 > 0$$
$$(x + 4)(x - 1) > 0$$

(*a*) Here we have two factors $(x + 4)$ and $(x - 1)$ that have a product that is positive.

(*b*) To have a positive product we must have values of x that give values to the two factors that have the same sign.



$\therefore$ S.S. is $\{x | x < -4 \text{ or } x > 1, x \in R\}$.

Exercise 6.5

1. By factoring first and then graphing the sign of each factor on a real number line, solve the following.

(*a*)	$x^2 - 5x + 6 > 0$	(*f*)	$x^2 - 4x - 5 < 0$
(*b*)	$x^2 - 6x - 7 < 0$	(*g*)	$x^2 - x - 6 < 0$
(*c*)	$x^2 + 4x + 3 < 0$	(*h*)	$x^2 + x - 12 > 0$
(*d*)	$x^2 + 5x - 14 > 0$	(*i*)	$x^2 + 7x + 10 < 0$
(*e*)	$x^2 - 7x + 12 > 0$	(*j*)	$x^2 - 2x - 8 > 0$

1.(a) $\{x | x < 2 \text{ or } x > 3\}$
(d) $\{x | x < -7 \text{ or } x > 2\}$
(g) $\{x | -2 < x < 3\}$
(j) $\{x | x < -2 \text{ or } x > 4\}$

Puzzles and Problems

1. Paul and Anne decided to solve the problem
$x^2 + 2x + 1 = x^2 - 1$.

Paul said:

"If $x^2 + 2x + 1 = x^2 - 1$
then $2x + 1 = -1$
and $2x = -2$
$\therefore x = -1$"

Anne said:

"If $x^2 + 2x + 1 = x^2 - 1$
then $(x + 1)^2 = (x - 1)(x + 1)$
$\therefore \dfrac{(x+1)^2}{(x+1)} = \dfrac{(x-1)(x+1)}{(x+1)}$
$\therefore x + 1 = x - 1$
$\therefore 0 = 2$!"

What went wrong? Can you help Anne and Paul with their problem?

Don't ask me why!

2. Having solved Question 1 finally, Anne and Paul tried this one.
Solve $x^2 + 2x + 1 = 4$.

Paul said:

"If $x^2 + 2x + 1 = 4$
$(x + 1)^2 = 2^2$
$\therefore x + 1 = 2$
and $x = 1$"

Anne said:

"If $x^2 + 2x + 1 = 4$
then $x^2 + 2x - 3 = 0$
$\therefore (x + 3)(x - 1) = 0$
$\therefore x + 3 = 0$ or $x - 1 = 0$
$\therefore x = -3$ or $x = 1$"

Who is correct? Do you think they'll ever get along?

Unit 7
Equations and Inequations – Two Variables

7.1 Relations Revisited

A You will recall from *Mathematics for a Modern World, Book 2* that a relation in mathematics can be expressed in many ways.

(*a*) In words:

"A number is 2 more than twice another number." ← Statement

(*b*) By a defining sentence:

$$y = 2x + 2$$ ← Equation

(*c*) By a set of ordered pairs:

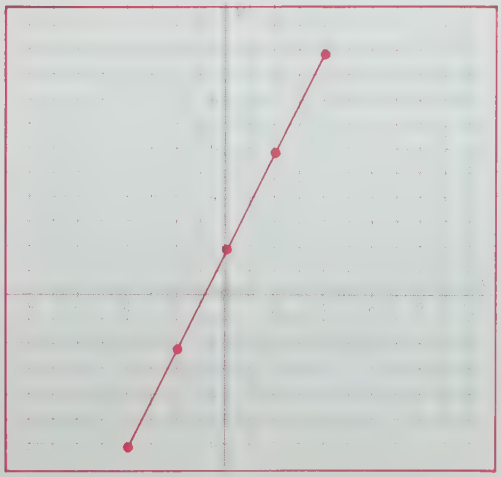
$\{(0, 2), (2, 6), (-2, -2), (4, 10), (-4, -6)\}$ ← Ordered Pairs

(*d*) By a table of values:

x	0	2	-2	4	-4
y	2	6	-2	10	-6

← Table of Values

(*e*) By a graph:



← Graph

FIRST DEGREE RELATION
$3x + 2y = 7$

SECOND DEGREE RELATION
$4x^2 + 9y^2 = 36$

A first degree equation with two variables is referred to as a *linear equation* because its graph is a straight line.

1. Which of the following are linear equations?
   (*a*) $x^2 + y^2 = 25$ (*c*) $4x^2 - 9y^2 = 36$
   (*b*) $3x + 2y = 7$ (*d*) $y = 3x - 2$

2. Which of the relations defined by the following equations has a straight line graph?
   (*a*) $3x - 2 = y$ (*d*) $5x - y = 4$
   (*b*) $4x^2 - y = 2$ (*e*) $x - 5y = 7$
   (*c*) $y = 3x^2 - 4$ (*f*) $4x^2 + 4y^2 = 25$

But a third point should be found as a check.

B To graph a linear equation you need only two points, and you can get two points easily by first letting x be zero and then y be zero.

Example 1

Graph the relation given by $3x + 2y = 12$.

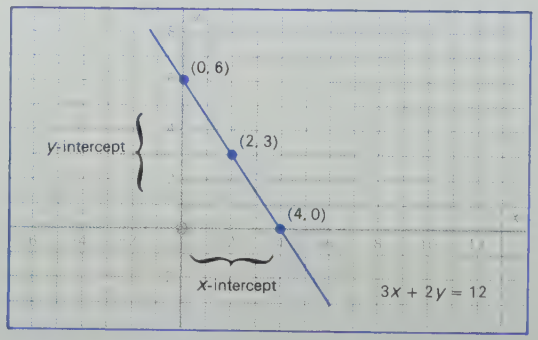
Solution

Let $x = 0$, $\therefore 2y = 12$
$y = 6$ ← THE y-INTERCEPT
Let $y = 0$, $\therefore 3x = 12$
$x = 4$ ← THE x-INTERCEPT

$\therefore$ the points (0, 6) and (4, 0) are on the graph.

Check

When $x = 2$
$3(2) + 2y = 12$
$6 + 2y = 12$
$2y = 6$
$y = 3$
$\therefore$ (2, 3) should be on the graph too.



Example 2

Graph the relation given by $5x + y = 0$.

Solution

Let $x = 0$, $\therefore y = 0$

Let $y = 0$ $\therefore 5x = 0$

$x = 0$

} Both the intercepts are zero! We get only one point!

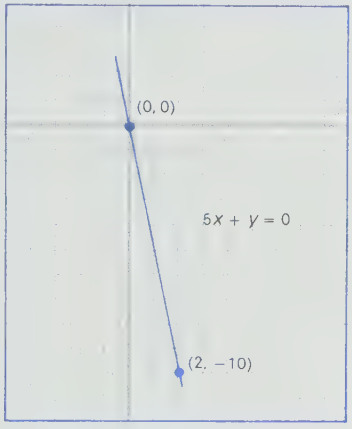
When both intercepts are zero, you must select another value for either x or y.

Let $x = 2$, $\therefore 5(2) + y = 0$

$10 + y = 0$

$y = -10$

$\therefore (0, 0)$ and $(2, -10)$ are on the graph.



Exercise 7.1

Written Solutions

1. Calculate the intercepts for the following lines.
   (*a*) $x + y = 3$
   (*b*) $2x + y = 6$
   (*c*) $3x + 5y = 15$
   (*d*) $x - y = 4$
   (*e*) $2x - 3y = -18$
   (*f*) $3x + 15 = y$
   (*g*) $2x - 6 = -3y$
   (*h*) $24 - 3x = 8y$
   (*i*) $2y - 3x = 30$
   (*j*) $4y - 5x = -40$

2. Graph the linear relations defined in the following.
   (*a*) $5x + 2y = 20$
   (*b*) $3x - 5y = 15$
   (*c*) $x - 7y = 14$
   (*d*) $3x + 2y = -6$
   (*e*) $5x + 2y = 0$
   (*f*) $3x = 4y$
   (*g*) $2x + 10 = 3y$
   (*h*) $15x + 30 = -2y$

3. Select the linear relations from those listed below and graph them.
   (*a*) $3x + y^2 = 6$
   (*b*) $4x - y = 7$
   (*c*) $3x - 2y = x^2$
   (*d*) $11x + 7y = -77$
   (*e*) $3x^2 - 4y^2 = 12$
   (*f*) $9y - 27 = 3x$
   (*g*) $5x + 15 = 3y$
   (*h*) $3y^2 - 4x = 6$

1.(b) 3, 6 (c) 5, 3 (g) 3, 2 (h) 8, 3

4. Graph the following linear relations and examine their similarities.
   (*a*) $y = 3x + 7$
   (*b*) $y = 3x - 6$
   (*c*) $y = 3x + 15$
   (*d*) $3x - y = -5$

5. Graph the following linear relations. What do the graphs have in common?
   (*a*) $y = 2x + 9$
   (*b*) $3x + y = 9$
   (*c*) $3y - 27 = 5x$
   (*d*) $y = 5x + 9$

6. (*a*) Graph $(y - x)(y - 2x) = 0$ by completing the table of values below.

x	1	1	2	2	3	3	-2	-2	-3	-3	0
y	1	2									

   (*b*) Show that the graph in (*a*) consists of those points on the graph of either $y - x = 0$ or $y - 2x = 0$.

7. Graph the following.
   (*a*) $(x - 5y)(x - 2y) = 0$
   (*b*) $(y - x + 1)(y + x - 5) = 0$
   (*c*) $(y - 3x)(y + 2x) = 0$
   (*d*) $(y - 3)(y + 1) = 0$
   (*e*) $(y - 2x + 1)(y + x + 1) = 0$
   (*f*) $(x - 2)(x + 3) = 0$

7.2 Intersections and Unions – Two Variables

AND

"Points in common"

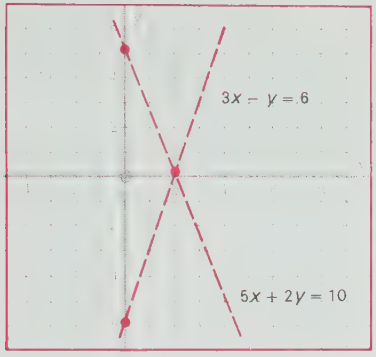
A *Intersections*

Graph $\{(x, y) \mid 5x + 2y = 10 \text{ AND } 3x - y = 6\}$.

THE SET OF ORDERED PAIRS SUCH THAT

Consider $5x + 2y = 10$.
Let $x = 0$, $\therefore 5(0) + 2y = 10$
$2y = 10$
$y = 5$
Let $y = 0$, $\therefore 5x + 2(0) = 10$
$5x = 10$
$x = 2$
Points are (0, 5) and (2, 0).

Consider $3x - y = 6$.
Let $x = 0$, $\therefore 3(0) - y = 6$
$-y = 6$
$y = -6$
Let $y = 0$, $\therefore 3x = 6$
$x = 2$
Points are (0, −6) and (2, 0).



The only point on $5x + 2y = 10$ *and* $3x - y = 6$ is the point $(2, 0)$.

$\therefore \{(x, y) \mid 5x + 2y = 10 \text{ AND } 3x - y = 6\} = \{(2, 0)\}$

Why are both lines dotted?

B *Unions*

Graph $\{(x, y) \mid 7x + 2y = 14 \text{ OR } 5x - 2y = -20\}$.

THE SET OF — ORDERED PAIRS — SUCH THAT

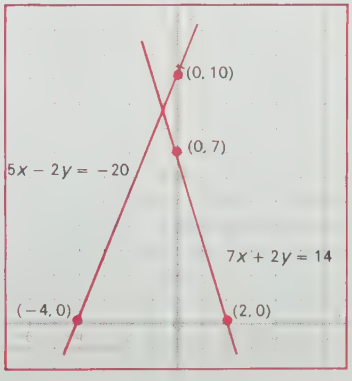
OR

"Points satisfying either".

Consider $7x + 2y = 14$
Let $x = 0$, $\therefore 2y = 14$
$y = 7$
Let $y = 0$, $\therefore 7x = 14$
$x = 2$
Points are $(0, 7)$ and $(2, 0)$.

Consider $5x - 2y = -20$
Let $x = 0$, $\therefore -2y = -20$
$y = 10$
Let $y = 0$, $\therefore 5x = -20$
$x = -4$
Points are $(0, 10)$ and $(-4, 0)$.

This is the required graph.



Why are both lines solid?

Exercise 7.2

1. Graph the solution set for each of the following.
   (*a*) $\{(x, y) \mid x + y = 2 \text{ AND } 2x - y = -4\}$
   (*b*) $\{(x, y) \mid 4x + y = 18 \text{ AND } 2x - y = 12\}$
   (*c*) $\{(x, y) \mid 5x + y = -10 \text{ OR } 3x + y = -6\}$
   (*d*) $\{(x, y) \mid x - y = 5 \text{ AND } x - y = 0\}$
   (*e*) $\{(x, y) \mid 4x + 7y = 28 \text{ OR } x - 3y = 9\}$
   (*f*) $\{(x, y) \mid 2x - 3y = -5 \text{ AND } 5x + 3y = 19\}$
   (*g*) $\{(x, y) \mid 5x - 4y = 20 \text{ OR } x = -y\}$
   (*h*) $\{(x, y) \mid 8x - y = 24 \text{ AND } 2x + y = 11\}$
   (*i*) $\{(x, y) \mid 4x - y = 6 \text{ OR } 3x - 2y = 7\}$
   (*j*) $\{(x, y) \mid 2x + 3y = 2 \text{ AND } 4x - 3y = 14\}$
   (*k*) $\{(x, y) \mid 3x - 5y = -15 \text{ OR } x + 7y = 7\}$
   (*l*) $\{(x, y) \mid x - 5y = 0 \text{ AND } x + 3y = 0\}$

7.3 Intersections by Algebra

Graphical methods for determining intersections of graphs are quite valid but can be inaccurate when the solution set contains numbers other than integers. The three sections following illustrate the algebraic methods for solving for the point of intersection of two linear graphs.

Addition - Subtraction Method

A The examples below review the key points of the addition-subtraction algebraic method.

Example 1

Solve algebraically.

$$2x + y = 7$$
$$3x - y = 8$$

Solution

$$2x + y = 7 \quad (1)$$
$$3x - y = 8 \quad (2)$$

Add $5x \qquad = 15$

$\therefore x = 3$

Substitute $x = 3$ in (1)

$2(3) + y = 7$

$6 + y = 7$

$y = 1$

Check

If $x = 3, y = 1$

in $2x + y = 7$

L.S. $= 2(3) + 1$
$= 6 + 1$
$= 7$

R.S. $= 7$

in $3x - y = 8$

L.S. $= 3(3) - 1$
$= 9 - 1$
$= 8$

R.S. $= 8$

$\therefore x = 3, y = 1$

Example 2

Solve algebraically.

$$2x - y = -7$$
$$x - 3y = 4$$

Solution

$$\begin{aligned} 2x - y &= -7 \quad (1) \\ x - 3y &= 4 \quad (2) \end{aligned}$$

$$2x - y = -7 \quad (1)$$

(2) × 2 $\quad 2x - 6y = 8 \quad (3)$

Subtract $\quad 5y = -15$

$\therefore y = -3$

Substitute $y = -3$ in (1)

$$2x - (-3) = -7$$
$$2x + 3 = -7$$
$$2x = -7 - 3$$
$$2x = -10$$
$$x = -5$$

STEPS

1 Multiply one equation, or both, by a suitable factor so that addition or subtraction will eliminate one variable.

2 Solve for the remaining variable.

3 Substitute in one of the original equations to find the other variable.

Check

If $x = -5, y = -3$

in $2x - y = -7$

L.S. $= 2(-5) - (-3)$
$= -10 + 3$
$= -7$

R.S. $= -7$

in $x - 3y = 4$

L.S. $= -5 - 3(-3)$
$= -5 + 9$
$= 4$

R.S. $= 4$

$\therefore x = -5, y = -3$

Example 3

Solve algebraically.

$$\frac{x}{8}+\frac{y}{3}=15$$

$$\frac{x}{4}-\frac{y}{5}=4$$

Solution

$$\frac{x}{8}+\frac{y}{3}=15 \qquad (1)$$

$$\frac{x}{4}-\frac{y}{5}=4 \qquad (2)$$

Why multiply (1) by 24 and (2) by 20?

(1) $\times 24$ $\quad 24\left(\frac{x}{8}\right)+24\left(\frac{y}{3}\right)=15(24) \qquad (3)$

(2) $\times 20$ $\quad 20\left(\frac{x}{4}\right)-20\left(\frac{y}{5}\right)=4(20) \qquad (4)$

$$3x+8y=360 \qquad (3)$$

$$5x-4y=80 \qquad (4)$$

Why multiply (4) by 2?

$$3x+8y=360 \qquad (3)$$

(4) $\times 2$ $\quad 10x-8y=160 \qquad (5)$

Add $\quad 13x=520$

$$\therefore x=40$$

Substitute $x=40$ in (1)

$$\frac{40}{8}+\frac{y}{3}=15$$

$$5+\frac{y}{3}=15$$

$$\frac{y}{3}=10$$

$$3\left(\frac{y}{3}\right)=3(10)$$

$$\therefore y=30$$

$$\therefore x=40,\ y=30$$

Remember to check!

B *Substitution Method*

Example 1

Solve algebraically.

$$2x - 3y = 7 \quad (1)$$
$$3x + 2y = 4 \quad (2)$$

Solution

From (1), $2x = 3y + 7$

$$\therefore x = \frac{3y + 7}{2}$$

Substitute in (2) $3\left(\frac{3y + 7}{2}\right) + 2y = 4$

Multiply by 2. $3(3y + 7) + 4y = 8$

$$9y + 21 + 4y = 8$$
$$13y = -13$$
$$y = -1$$

Substitute $y = -1$ in $x = \frac{3y + 7}{2}$

$$\therefore x = \frac{3(-1) + 7}{2}$$
$$= \frac{-3 + 7}{2}$$
$$\therefore x = 2$$

$$\boxed{\therefore x = 2, y = -1}$$

Example 2

Solve algebraically.

$$2x + y = -1 \quad (1)$$
$$5x + 6y = 8 \quad (2)$$

Solution

From (1), $y = -2x - 1$

Substitute in (2) $5x + 6(-2x - 1) = 8$

$$5x - 12x - 6 = 8$$
$$-7x = 14$$
$$x = -2$$

Substitute $x = -2$ in $y = -2x - 1$

$$y = -2(-2) - 1$$
$$= 3$$

$$\boxed{\therefore x = -2, y = 3}$$

Substitution Method

STEPS

1 Solve one equation for one variable.

2 Substitute in the other equation.

3 Solve for the value of one variable.

4 Substitute to find the second variable.

Remember to check!

Comparison Method

C *Comparison Method*

Example

Solve algebraically.

$$3x + 2y = 8 \quad (1)$$
$$2x + 5y = 31 \quad (2)$$

Solution

1 Solve both equations for the same variable.

From (1), $3x = 8 - 2y$ and from (2), $2x = 31 - 5y$

$$\therefore x = \frac{8 - 2y}{3} \qquad x = \frac{31 - 5y}{2}$$

2 Equate the two quantities in (1)

$$\therefore \frac{8 - 2y}{3} = \frac{31 - 5y}{2}$$

$$\frac{{}^{2}\cancel{6}(8 - 2y)}{\cancel{3}_{1}} = \frac{{}^{3}\cancel{6}(31 - 5y)}{\cancel{2}_{1}}$$

$$16 - 4y = 93 - 15y$$

3 Solve for the one variable.

$$11y = 77$$
$$y = 7$$

4 Substitute to find the second variable.

Substitute in $x = \frac{8 - 2y}{3}$

$$\therefore x = \frac{8 - 2(7)}{3}$$
$$= \frac{8 - 14}{3}$$
$$\therefore x = -2$$

Check!

$$\therefore \quad x = -2, y = 7$$

Exercise 7.3

Written Solutions

1. By adding or subtracting, eliminate one variable and then complete the solution.

(a) $5m + 2n = 19$
$3m + 2n = 13$

(b) $4x + y = 18$
$2x - y = 12$

(c) $7x - 3y = 15$
$4x - 3y = 9$

(d) $4w + 3t = 9$
$4w + t = 11$

(e) $5u - 2v = 4$
$4u + 2v = \frac{1}{2}$

(f) $6b + 5c = 22$
$-6b - 4c = -20$

(g) $-5m + 16n = 19$
$-5m + 12n = 27$

(h) $2x - 5n = -3\frac{1}{2}$
$-2x + 3n = 2\frac{1}{2}$

Same signs - subtract
Opposite signs - add

1.(a) (3, 2) (b) (5, −2) (e) $(\frac{1}{2}, -\frac{3}{4})$ (f) (2, 2)

2. Solve the following equations using the method of addition or subtraction.

(*a*) $4x + 3y = -2$
$2x - 3y = -10$

(*b*) $2x - 5y = 17$
$3x - y = 6$

(*c*) $2x - y = -5$
$x + 3y = 1$

(*d*) $2x - 5y = 1$
$7x + 3y = 24$

(*e*) $6r - 7s = 17$
$7r - 8s = -18$

(*f*) $8x - y = 34$
$x + 8y = 53$

3. Solve using the substitution method.

(*a*) $x + 2y = 18$
$2x + 5y = 41$

(*b*) $7x + 2y = 47$
$5x - 4y = 1$

(*c*) $4a + 3b = -1$
$5a + 3b = 1$

(*d*) $x + 4y = 1$
$3x - 8y = 33$

(*e*) $a + b = 4$
$2a - 3b = 13$

(*f*) $2x + 3y = -15$
$3x - 5y = 63$

4. Solve using the comparison method.

(*a*) $a = 5 - b$
$3a = b + 7$

(*b*) $x + 2y = 8$
$x + y = 5$

(*c*) $5a + b = 11$
$3a - 8b = 41$

(*d*) $3x - 5y = 1$
$4x + y = 9$

(*e*) $2x + y = 4$
$4x + 3y = 6$

(*f*) $3m - 4n = 16$
$7m + 3n = 62$

5. Solve the following equations. (Use any method.)

Eliminate the fractions first.

(*a*) $\frac{a}{4} + \frac{b}{2} = 8$
$\frac{a}{2} - \frac{b}{4} = 1$

(*b*) $\frac{2x}{9} + \frac{3y}{7} = 13$
$\frac{5x}{3} + y = 51$

(*c*) $\frac{2x}{5} - \frac{3y}{2} = -5$
$\frac{3x}{2} - \frac{2y}{3} = 11$

(*d*) $\frac{1}{2}x - \frac{1}{5}y = 4$
$\frac{1}{7}x + \frac{1}{15}y = 3$

(*e*) $\frac{u}{2} + \frac{v}{3} = 2\frac{1}{6}$
$\frac{u}{2} - \frac{v}{3} = \frac{5}{6}$

(*f*) $\frac{x}{5} - \frac{y}{4} = 0$
$3x - \frac{1}{2}y = 52$

(*g*) $a - b = 6$
$\frac{2}{3}a - \frac{3}{5}b = 3$

(*h*) $10x + 2y = 30$
$\frac{5x}{2} + \frac{3y}{5} = 8$

2.(b) (1, −3) (e) (10, 11)

3.(a) (8, 5) (d) $7, -\frac{3}{2}$)

4.(c) (3, −4) (f) (8, 2)

5.(a) (8, 12) (e) (3, 2)
(d) (14, 15) (h) (2, 5)

7.4 Inequations Involving AND and OR

A You will recall from *Mathematics for a Modern World, Book 2* that an inequation involving two variables represents a *region* when plotted.

The equation represents the boundary line between < and >.

Example

Graph $4x - 9y \geqslant 36$.

Solution

Consider the boundary line $4x - 9y = 36$.

Let $x = 0$, $\therefore -9y = 36$

$y = -4$

Let $y = 0$, $\therefore 4x = 36$

$x = 9$

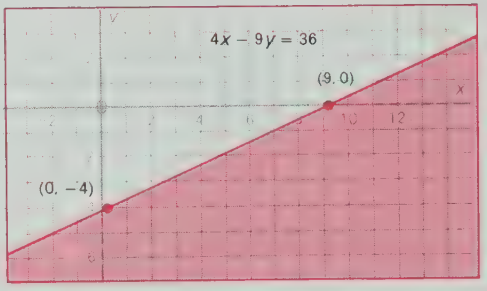
$\therefore$ points are $(0, -4)$ and $(9, 0)$.

Check $(0, 0)$ in $4x - 9y > 36$

$0 - 0 > 36$

$0 > 36$, which is false.

$\therefore (0, 0)$ is not in required region.



Why is the line solid?

B We will now look at the intersection of two such regions.

Example 1

Graph $\{(x, y) \mid 5x + 2y \leqslant 10 \text{ AND } 3x - 4y \geqslant 12\}$.

Solution

* These are the boundary lines

	*		*
Consider	$5x + 2y = 10$	and	$3x - 4y = 12$
Let $x = 0$,	$\therefore 2y = 10$		$-4y = 12$
	$y = 5$		$y = -3$
Let $y = 0$,	$\therefore 5x = 10$		$3x = 12$
	$x = 2$		$x = 4$
$\therefore$ points are	$(0, 5)$ and $(2, 0)$		$(0, -3)$ and $(4, 0)$

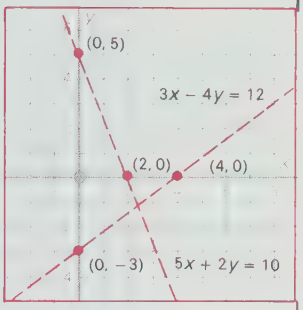
Check (0, 0) in $5x + 2y < 10$

$0 + 0 < 10$

$0 < 10$

true

$\therefore$ region is below $5x + 2y = 10$



Check (0, 0) in $3x - 4y > 12$

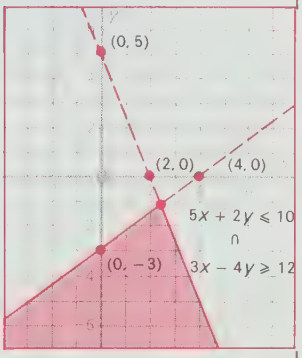
$0 - 0 > 12$

$0 > 12$

false

$\therefore$ region is below $3x - 4y = 12$

$\therefore$ required region is below $5x + 2y = 10$
and below $3x - 4y = 12$.



Why are parts of the lines solid?

Why is this a solid dot?

Example 2

Graph $\{(x, y) \mid 5x + y < 10 \text{ AND } 2x - 5y \leqslant 20\}$.

Solution

Consider $5x + y = 10$ and $2x - 5y = 20$*

Let $x = 0$, $\therefore y = 10$ $\quad -5y = 20$

$y = -4$

Let $y = 0$, $\therefore 5x = 10$ $\quad 2x = 20$

$x = 2$ $\quad x = 10$

$\therefore$ points are (0, 10), (2, 0) $\quad$ (0, −4), (10, 0)

* Again, the boundary lines first.

Check (0, 0) in $5x + y < 10$

$0 + 0 < 10$

$0 < 10$

true

$\therefore$ region is below $5x + y = 10$

Check (0, 0) in $2x - 5y < 20$

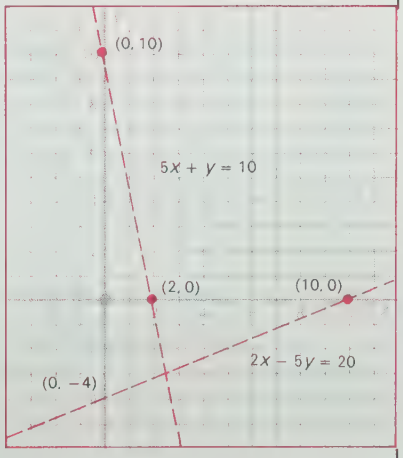
$0 < 20$

true

$\therefore$ region is above $2x - 5y = 20$

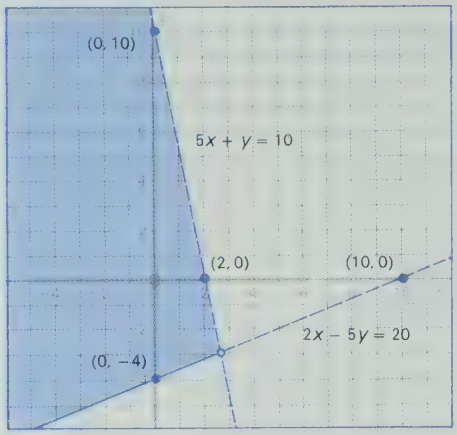
$\therefore$ required region is below $5x + y = 10$

and above $2x - 5y = 20$.



Dot the lines first. Then check (0,0) in each equation. Then shade in the regions.

Why is part of $2x - 5y = 20$ solid?



Is the point of intersection included?

C We now look at the union of two regions.

Example

Graph $\{(x, y) \mid 4x - 7y < 14 \text{ OR } 5x \geqslant y\}$.

Solution

Get the boundary lines.

Consider $4x - 7y = 14$ and $5x = y$

Let $x = 0$, $\therefore -7y = 14$ $\quad$ $0 = y$

$y = -2$ $\quad$ $y = 0$

Let $y = 0$, $\therefore 4x = 14$ $\quad$ $5x = 0$

$x = \frac{14}{4}$ $\quad$ $x = 0$

$x = \frac{7}{2}$ $\quad$ let $x = 1$ $\quad$ $5 = y$

$y = 5$

$\therefore$ points are $(0, -2)$ $\left(\frac{7}{2}, 0\right)$ $\quad$ $(0, 0)$ $\quad$ $(1, 5)$

Graph the lines (dotted).

Check $(0, 0)$ in $4x - 7y < 14$

$0 < 14$

true

Check each region.

$\therefore$ region is above $4x - 7y = 14$

Why not use (0,0) this time?

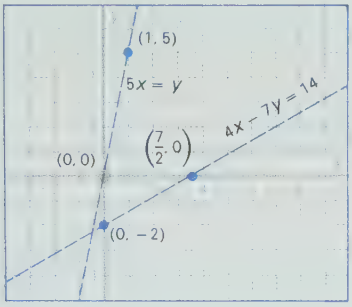
Check $(4, 0)$ in $5x > y$

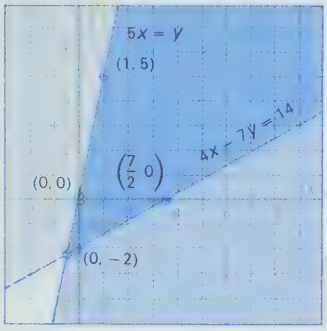
$20 > 0$

true

$\therefore$ region is below $5x = y$

$\therefore$ required region is above $4x - 7y = 14$ *or* below $5x = y$.





Make solid lines where necessary.

Why is the point of intersection included?

Exercise 7.4

1. Graph the inequations.
   (a) $\{(x, y) | 5x + 2y \geqslant 14\}$
   (b) $\{(x, y) | 3x - 7y \leqslant 21\}$
   (c) $\{(x, y) | x - 8y \leqslant 0\}$
   (d) $\{(x, y) | 3x - y > 7\}$
   (e) $\{(x, y) | 9x + y < 12\}$
   (f) $\{(x, y) | 3x + 11y < 11\}$
   (g) $\{(x, y) | 7x - y > 21\}$
   (h) $\{(x, y) | 15x - 2y < 0\}$
   (i) $\{(x, y) | 19x + y \geqslant 38\}$
   (j) $\{(x, y) | 14x - 3y \leqslant 14\}$

2. Graph the following intersections of regions.
   (a) $\{(x, y) | 4x - 3y \leqslant 24 \text{ AND } 3x + y \geqslant 6\}$
   (b) $\{(x, y) | x + 7y \leqslant 7 \text{ AND } 5x - y \geqslant 5\}$
   (c) $\{(x, y) | 7x - 2y > 14 \text{ AND } 2x + 3y > 12\}$
   (d) $\{(x, y) | 5x + y \leqslant 15 \text{ AND } 3x \leqslant 2y\}$
   (e) $\{(x, y) | 9x - 7y < 9 \text{ AND } 6x + y > 6\}$

3. Graph the following unions of regions.
   (a) $\{(x, y) | 11x + 2y < 11 \text{ OR } 3x + y > 6\}$
   (b) $\{(x, y) | 13x - 2y > 13 \text{ OR } x + y > 7\}$
   (c) $\{(x, y) | 2x - 3y \leqslant 0 \text{ OR } 5x + 2y \leqslant 10\}$
   (d) $\{(x, y) | 7x + 4y \geqslant 14 \text{ OR } x + 7y \leqslant 1\}$
   (e) $\{(x, y) | 5x - 3y < 0 \text{ OR } 2x + 5y > 1\}$

4. Graph the following regions.
   (a) $\{(x, y) | x + 6y < 18 \text{ AND } 2x - y < 6\}$
   (b) $\{(x, y) | 5x + y > 10 \text{ OR } 3x - 2y \leqslant 12\}$
   (c) $\{(x, y) | 7x - 3y \leqslant 7 \text{ OR } 2x + 5y \geqslant 5\}$
   (d) $\{(x, y) | 9x + 7y \geqslant 7 \text{ AND } 5x - y > 5\}$
   (e) $\{(x, y) | 3x - 7y \leqslant 18 \text{ OR } x + 11y < 11\}$
   (f) $\{(x, y) | 7x + 2y > 15 \text{ OR } x - y < 0\}$
   (g) $\{(x, y) | 11x + 2y < 0 \text{ AND } x - 7y \geqslant 0\}$

STEPS

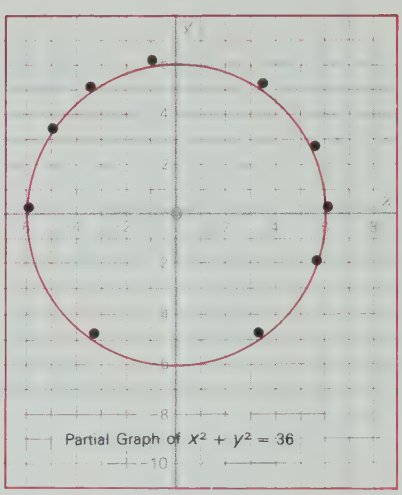
1 Boundary lines (dotted).
2 Determine which side of each line.
3 Check whether AND or OR.
4 Solid lines where necessary.

7.5 Some Non-Linear Graphs

For non-linear graphs, get "enough" points to make the shape clear.

A *Example*

Graph $\{(x, y) | x^2 + y^2 = 36\}$.



Solution

If $x = 0$, $y^2 = 36$

$y = \pm 6$

If $x = \pm 2$, $4 + y^2 = 36$

$y^2 = 32$

$y = \pm\sqrt{32}$

$\doteq \pm 5.7$

* Let the variable be + or − if it is squared in the relation.

If $x = \pm 4$, $16 + y^2 = 36$

$y^2 = 20$

$y = \pm\sqrt{20}$

$\doteq \pm 4.5$

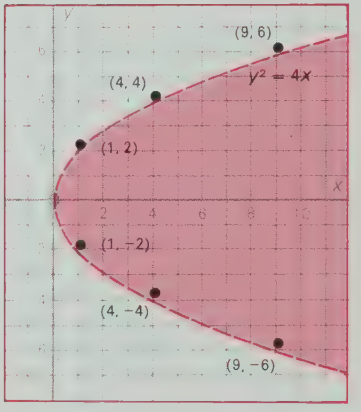
If $x = \pm 6$, $36 + y^2 = 36$

$y^2 = 0$

$y = 0$

B *Example*

Graph $\{(x, y) | y^2 < 4x\}$.



Solution

Consider $y^2 = 4x$

x	0	1	4	9
y	0	± 2	± 4	± 6

Can you see why $x = 0, 1, 4, 9, \cdots$ were used in $y^2 = 4x$?

If $x = 9$ $\quad y^2 = 36$

$y = \pm 6$

$\therefore$ points are $(0, 0)$, $(1, \pm 2)$, $(4, \pm 4)$, $(9, \pm 6)$

Any other point in the same region would give a similar result. Try it!

Check $(2, 0)$ in $y^2 < 4x$

$0 < 4(2)$

$0 < 8$ true

$\therefore$ $(2, 0)$ is in the required region.

Exercise 7.5

1. Graph the following non-linear relations.
   (*a*) $\{(x, y) | x^2 + y^2 = 25\}$ (*f*) $\{(x, y) | x^2 + y^2 = 49\}$
   (*b*) $\{(x, y) | y^2 = x\}$ (*g*) $\{(x, y) | 3x^2 + 4y^2 = 12\}$
   (*c*) $\{(x, y) | x^2 + y^2 = 4\}$ (*h*) $\{(x, y) | x^2 = 4y\}$
   (*d*) $\{(x, y) | 9x^2 + 4y^2 = 36\}$ (*i*) $\{(x, y) | y^2 = 4x + 4\}$
   (*e*) $\{(x, y) | y^2 = 2x\}$ (*j*) $\{(x, y) | y^2 = x + 3\}$

2. Graph the regions given by the following non-linear relations.
   (*a*) $\{(x, y) | x^2 + y^2 < 25\}$ (*f*) $\{(x, y) | y^2 > x\}$
   (*b*) $\{(x, y) | y^2 < 4x\}$ (*g*) $\{(x, y) | x^2 + y^2 > 4\}$
   (*c*) $\{(x, y) | 4x^2 + 9y^2 < 36\}$ (*h*) $\{(x, y) | 3x^2 + 4y^2 > 12\}$
   (*d*) $\{(x, y) | x^2 \geqslant 4y\}$ (*i*) $\{(x, y) | y^2 \leqslant 8x\}$
   (*e*) $\{(x, y) | x^2 + y^2 \geqslant 16\}$ (*j*) $\{(x, y) | x^2 + y^2 < 1\}$

3. Graph the following to show the required intersections.
   (*a*) $\{(x, y) | 5x - 2y = 10 \text{ AND } y^2 = 4x\}$
   (*b*) $\{(x, y) | 3x + 2y = 6 \text{ AND } x^2 + y^2 = 25\}$
   (*c*) $\{(x, y) | 5x + y = 5 \text{ AND } y^2 = 2x\}$
   (*d*) $\{(x, y) | 3x - y = 9 \text{ AND } x^2 + y^2 = 36\}$
   (*e*) $\{(x, y) | x^2 = 4y \text{ AND } 7x - 2y = 7\}$
   (*f*) $\{(x, y) | x^2 + y^2 = 25 \text{ AND } x^2 + 9y^2 = 36\}$

4. The cross-sectional view of the retractable headlight of an experimental car is determined by the region enclosed by
   $$\{(x, y) | y^2 \leqslant 4x \text{ AND } 2x + y - 12 \leqslant 0\}$$
   Graph the region. Sketch your version of the side view of the front of the car.

5. A stone is thrown into a lake and the ripples formed move out and contact the shore on one side. After a short interval, the largest circle formed by the ripples is given by the equation $x^2 + y^2 = 49$. If the shoreline can be represented by the equation $3x - 5y = 15$, sketch the region of the lake that would be covered by ripples.

6. A radio station is allowed to broadcast its signal within a specified area. If the specified area for station MATH is given by $\{(x, y) | y^2 \leqslant 4x \text{ AND } x^2 + y^2 \leqslant 64\}$, graph the region covered by the station.

Unit 8
Problem Solving

8.1 Translating English to Algebra

There are many kinds of "problems", aren't there?

How to pass math?
How to pass your year?
How to meet the student in the next row?
How to earn enough to buy that dune buggy?
How to get that new dress?

And there are many ways to solve problems.

Ask other people. (That is, use the benefit of other people's experience.)
Trial and error (guessing). Computers often use this technique to solve problems.
Hard work. (That's for most of us.)
Use your brains. (The lazy man's way.)

Algebraic techniques are often needed in the "using your brains" method. The main problem with this technique is translating the facts into algebraic statements.

The problem must be approached much like translating to another language. In this case we are translating from English to algebra.

Let the number be

$x + 6$ or "6 more than the number of cars owned by Andy Granatelli."

Warm-Up Exercise

1. Translate the English phrases into algebraic expressions.
   (*a*) a number increased by 2
   (*b*) 2 less than a number
   (*c*) 5 times a number
   (*d*) 7 more than a number
   (*e*) his age decreased by 15
   (*f*) the speed doubled
   (*g*) the price divided by 7
   (*h*) twice a number increased by 7
   (*i*) 6 less than 5 times a number

2. Translate the algebraic expressions into English phrases.
   (*a*) $x + 6$
   (*b*) $y - 7$
   (*c*) $12 - x$
   (*d*) $5x + 7$
   (*e*) $4y - 12$
   (*f*) $24 - 3x$
   (*g*) $14 + 12x$
   (*h*) $\frac{x}{6} + 3$
   (*i*) $\frac{x+3}{6}$

3. Translate the following equations into English statements, and solve the equations.
   (*a*) $x + 3 = 7$
   (*b*) $y - 4 = 11$
   (*c*) $6 - x = 12$
   (*d*) $5x + 6 = 36$
   (*e*) $4z - 24 = 4$
   (*f*) $7x - 2 = 12$
   (*g*) $\frac{x}{3} - 3 = 1$
   (*h*) $\frac{a}{5} + 5 = 6$
   (*i*) $\frac{1}{3}(a - 1) = 2$
   (*j*) $3(x - 2) = 2(x + 5)$
   (*k*) $7x = x + 18$
   (*l*) $\frac{x}{2} + \frac{x}{3} = 5$

4. Translate the English statements into equations and solve them.
   (*a*) A number increased by 7 is 11.
   (*b*) A quantity decreased by 9 gives a result of 17.
   (*c*) Five times a number is 15.
   (*d*) Seven times a quantity decreased by 11 is 3.
   (*e*) A number divided by 3 gives a result of 7.

8.2 Problems With Two Unknowns

In problems of this type there must be enough information given to enable you to form two separate equations. To solve for two variables you need two equations.

Two variables — two equations.

A *Number Problems*

The sum of a number and twice a smaller number is 61. If the difference between the two numbers is 4, find the numbers.

Each sentence can be translated into an equation.

Solution

Let the larger number be x.
Let the smaller number be y.

$$x + 2y = 61 \quad (1)$$
$$x - y = 4 \quad (2)^*$$

Subtract $3y = 57$

$$y = 19$$

Substitute $y = 19$ in (2) $x - 19 = 4$

$$x = 23$$

$\therefore$ the numbers are 23 and 19.*

* Why not $y - x$?

* Check: do they satisfy the problem?

B *Volume and Capacity Problems*

Two trucks can move 144 m³ of earth by taking 8 loads each. They can move 146 m³ if one truck takes 12 loads and the other takes 5 loads. What is the capacity of each truck?

Solution

Let the capacity of the two trucks be x and y m³.

$$8x + 8y = 144 \quad (1)^*$$
$$12x + 5y = 146 \quad (2)^*$$
$$(1) \times 5 \quad 40x + 40y = 720 \quad (3)$$
$$(2) \times 8 \quad 96x + 40y = 1168 \quad (4)$$

Subtract $-56x = -448$

$$x = \frac{448}{56}$$
$$= 8$$

Substitute $x = 8$ in (1) $8(8) + 8y = 144$

$$8y = 80$$
$$y = 10$$

$\therefore$ the trucks have capacities of 8 and 10 m³.*

* 8 loads each - a total of 144 m³.
* 12 loads for one, 5 loads for the other, a total of 146 m³.

* Check in the statement of the problem.

C *Money Problems*

A restaurant bill of \$3.10 is paid in quarters and dimes. If the number of quarters exceeded the number of dimes by 4, how many of each were used?

Solution

Let the number of quarters be x.
Let the number of dimes by y.

$25x$ IS THE NUMBER OF CENTS IN x QUARTERS

$10y$ IS THE NUMBER OF CENTS IN y DIMES

USE 310 BECAUSE IT'S IN CENTS

MORE QUARTERS THAN DIMES

$$25x + 10y = 310 \quad (1)$$

$$x - y = 4 \quad (2)$$

$(2) \times 10 \quad 10x - 10y = 40 \quad (3)$

$$25x + 10y = 310 \quad (1)$$

Add $\quad 35x = 350$

$$x = 10$$

Substitute $x = 10$ in (2)

$$\therefore 10 - y = 4$$

$$-y = -6$$

$$y = 6$$

$\therefore$ number of quarters is 10 and number of dimes is 6.

Exercise 8.2

Written Solutions

1. The sum of two numbers is 18. If one number is added to 4 times the other number, the result is 51. What are the numbers?
2. The sum of a number and 3 times a second number is 87. If the first number is subtracted from twice the second, the result is 28. What are the numbers?
3. Three times a number increased by 7 times a second number totals 42. If 5 times the first number decreased by 3 times the second number is 26, find the numbers.
4. If a certain number is increased by twice another number, the sum is 45. The first number is 7 times the second. What are the numbers?

2. 18, 23
4. 35, 5

5. The total mass of 2 nails and 7 bolts is found to be 122 g. If one nail and one bolt are added, the total mass becomes 148 g. What is the mass of each?

6. Nine trucks, some carrying 10 t loads and the others 6 t loads, are required to remove 70 t of earth. How many of each type are required?

7. The sum of two numbers is $\frac{12}{5}$. Twice the smaller increased by 3 times the larger is $\frac{31}{5}$. What are the numbers?

8. Donna and Pat bought a used car together for $450. If Donna paid $110 more than Pat, how much did each pay?

9. Laurie was cutting up 14 m of drapery material. She wanted first to cut the cloth so that one part exceeded the other part by 4 m. Where should she make the cut?

10. If 10 kg of nails and 2 hammers cost $15.50 and 3 kg of nails and 1 hammer cost $7.35, how much do nails cost per kilogram? How much is one hammer?

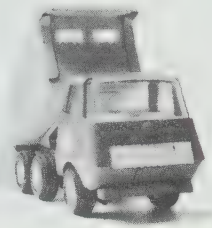
11. There once was a builder named Paul,
Who built, with his helper, a hall,
"n" dollars a week paid the two of them
And Paul got $\frac{5}{3}$ of his helper Clem,
How much each per week if you call?

12. Anne and Barb were able to sell 120 subscriptions to a magazine in one week. If Anne could have sold 15 more, she would have sold twice as many as Barb. How many did each sell?

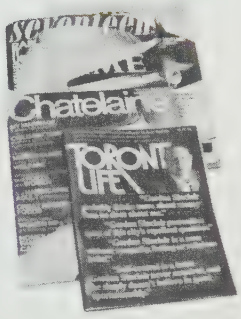
13. The mass of two defensive linemen together is 280 kg. Three times the lighter exceeds twice the heavier by 115 kg. What is the mass of each?

14. A student bought a 10-speed racer on sale for $92.00. She paid for the racer with $10 bills and $2 bills. If the number of $10 bills exceeded the number of $2 bills by 2, how many of each did she use?

15. Two groups of people entered a Drive-In Restaurant. The first group ordered 6 hamburgers and 5 orders of French fries, and paid $4.55. The second group ordered 10 hamburgers and 6 orders of French fries and paid $7.00. How much would you have to pay for 1 hamburger and 1 order of French fries?



t is the symbol for tonne.



6. 4 10 t trucks.
5 6 t trucks.

8. Donna paid $280.
Pat paid $170.

10. Nails cost 20¢/kg.
A hammer costs $6.75.

12. Anne sold 75.
Barb sold 45.

14. Number of $10 bills is 8.
Number of $2 bills is 6.

T. D. $s = 6$ points
F. G. $s = 3$ points

16. 11 touchdowns, 6 field goals
18. 9
20. $26.70

16. The leading scorer in a football league scored 84 points in a single season. He accumulated these points entirely by scoring touchdowns and field goals. If he scored 5 more touchdowns than he did field goals, how many of each did he score?

17. Find two numbers such that twice the larger decreased by 3 times the smaller is 19, and one-half the larger plus one-third the smaller is 8.

18. A restaurant bill of $2.85 was paid using quarters and dimes only. If twice the number of quarters is 6 less than 4 times the number of dimes, how many quarters were used?

19. The difference of two numbers is 17. Twice the larger number exceeds 3 times the smaller by 20. Find the numbers.

20. The local store is running a special on bacon and eggs. If one of your neighbors paid $30.00 for 10 kg of bacon and 3 dozen eggs, and another neighbor paid $48.00 for 20 kg of bacon and 2 dozen eggs, what should you pay for 7 kg of bacon and 4 dozen eggs?

8.3 More Problems With Two Unknowns

A *Age Problems*

The ages of a father and son total 51 years. In 9 years, the father will be twice as old as the son. How old is each now?

Solution

Let the present age of the father be x years and the son y years. This means that in 9 years the father will be $(x + 9)$ and the son will be $(y + 9)$.

$$x + y = 51 \quad (1) \qquad \text{FIRST SENTENCE}$$
$$(x + 9) = 2(y + 9) \quad (2) \qquad \text{SECOND SENTENCE}$$
$$\text{From } (2) \quad x + 9 = 2y + 18 \quad (2)$$
$$\therefore x - 2y = 9 \quad (2)$$
$$x + y = 51 \quad (1)$$
$$\text{Subtract} \quad -3y = -42$$
$$\therefore y = 14$$
$$\text{Substitute} \quad y = 14 \text{ in } (1)$$
$$\therefore x + 14 = 51$$
$$x = 37$$

$\therefore$ the father is 37 years and the son 14 years.

B *Perimeter Problems*

A rectangular room is 10 m longer than it is wide. If the perimeter of the room is 64 m, find the dimensions.

Make a sketch whenever you can.

Solution

Let the length of the room be x m and the width be y m.

$\therefore x = y + 10$ (1) ← *Why add to y?*

$2x + 2y = 64$ (2)

From (1) $x - y = 10$ (3)

Multiply (3) by 2 $2x - 2y = 20$ (4)

$2x + 2y = 64$ (2)

Add $4x = 84$

$\therefore x = 21$

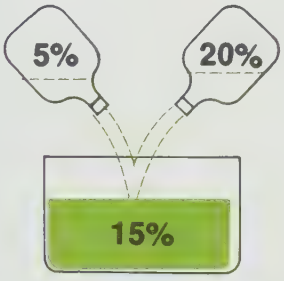
Substitute in (1) $21 = y + 10$

$\therefore y = 11$

$\therefore$ length of room is 21 m and width of the room is 11 m.

C *Mixture Problems*

A photographer uses two types of developer solution. One type is a 5% solution and the other is a 20% solution. For a particular job, she requires 30 ℓ of a 15% solution. How much of each should she mix?



Solution

Let the number of litres of the 5% solution be x. Let the number of litres of the 20% solution be y.

$x + y = 30$ (1) ← *Number of litres required.*

5% of x + 20% of y = 15% of 30 (2) ← *Amount of developer in each solution.*

From (2) $\frac{5}{100}x + \frac{20}{100}y = \frac{15}{100} \times 30$ (3)

(3) $\times$ 100 $5x + 20y = 450$ (4)

(4) $\div$ 5 $x + 4y = 90$ (5)

But from (1) $x + y = 30$ (1)

Subtract $3y = 60$

$y = 20$

Substitute in (1) $x + 20 = 30$

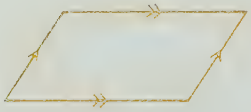
$\therefore x = 10$

$\therefore$ she should mix 10 ℓ of 5% solution and 20 ℓ of 20% solution.

Exercise 8.3

Written Solutions

1. A mother is 3 times as old as her daughter. In 11 years she will be just twice as old as her daughter. How old is each now?
2. Doug is twice as old as his brother. The sum of their ages is one-half the age of their father. If the father is 42 years old, how old are his sons?
3. Twice a girl's age added to her brother's age totals 36. If the boy is three years older than his sister, how old is each now?
4. Peter's father is 5 times as old as he is. In 21 years his father will just be twice as old as Peter. How old is each now?
5. Laurie is four years older than her brother. In 8 years the total of their ages will be 34. How old are they?
6. A mother is 25 years older than her daughter. In 10 years the total of their ages will be 67. Find their present ages.
7. The length of a rectangular garage is 6 m more than the width. If the perimeter of the garage is 40 m, find the dimensions.
8. The equal sides of an isosceles triangle are each 3 m longer than the base. If the perimeter of the triangle is 24 m, find the length of each side.
9. The longer sides of a parallelogram are 8 m longer than the shorter sides. If the perimeter is 64 m, find the dimensions of the parallelogram.
10. How much water and 50% antifreeze solution should be mixed to make 2 ℓ of a 20% solution?
11. A garage attendant tests the antifreeze in a car and finds that it has 10% antifreeze. After draining the radiator, how much pure antifreeze must be mixed with the present solution to make 35 ℓ of a 30% solution?
12. A science teacher had 8 ℓ of a 30% acid solution. She needed 10 ℓ of a 50% acid solution and decided to mix some of her 70% acid solution with some of the 30% solution in order to make it. How much of each should she mix?



Water is a 0% solution.

1. Mother's age is 33 years. Daughter's age is 11 years.
3. Girl is 11 years old. Brother is 14 years old.
5. Laurie's age is 11 years. Brother's age is 7 years.
7. Length = 13 m Width = 7 m
9. Length = 20 m Width = 12 m

13. Doug is 6 years older than Peter. In 9 years Peter will be $\frac{2}{3}$ the age of Doug. How old are they now?

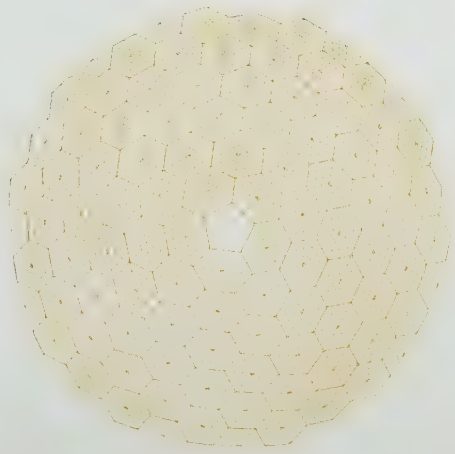
14. A gardener had a drum of pure weed killer and another drum of a 10% mixture of weed killer (in water). How much of each should he mix in order to form 80 ℓ of a 25% weed killer solution?

15. The cooling system of a car had a capacity of 60 ℓ. If the solution in the system contained 15% antifreeze, how much of this solution and how much pure antifreeze would have to be mixed in order to mix 60 ℓ of a 25% solution? How would you actually mix the solution?

16. A check of the developer solution at the SPEE-DEE Film Developing Company indicated that it was a 20% solution. How much of this liquid and how much of a 12% solution would have to be mixed to form 24 ℓ of a 15% solution?

17. A gardener with the ENDWEED Company wanted to mix 1 ℓ of a 30% weed control solution for a particular lawn. She has a 15% solution and a 50% solution already mixed. How many millilitres of each should she mix?

18. An air conditioning system should contain 50 ℓ of a salt solution called brine. In a particular air conditioner, the brine is to be 10% salt, but a check shows that it is in fact 14%. If the unit contains 50 ℓ of the 14% solution, how much should be drained and replaced by water to make 50 ℓ of a 10% solution?



Smart gardener!

So how much would they add?

13. Doug is 9 years old. Peter is 3 years old.

16. 9 ℓ of 20% solution with 15 ℓ of 12% solution.

18. $14\frac{2}{7}$ ℓ (or 14.3 ℓ) of 14% solution.

Maintaining Basic Skills 2

1. Solve the equations.

(a) $2x + 5 = 13$

(b) $3x - 7 = 11$

(c) $9 - 4x = 1$

(d) $5 - 3x = 14$

(e) $\frac{1}{4}y = 8$

(f) $\frac{2}{3}z = 7$

(g) $\frac{3w}{10} = \frac{12}{5}$

(h) $\frac{1}{3}y = 10 - \frac{y}{2}$

(i) $3 - 4t = -6$

(j) $6(x - 5) - 4(x - 3) = -22$

(k) $\frac{y - 4}{5} - \frac{1}{2}(y + 1) = -4$

(l) $\frac{x + 2}{3} = \frac{3x - 1}{6} - \frac{x + 5}{4}$

2. Solve for x.

(a) $px = q$

(b) $tx + 5 = wx$

(c) $\frac{x}{r} = s$

(d) $ax + b = c$

(e) $ax + b(x - 1) = 4$

(f) $\frac{x}{d} + e = 3x$

(g) $\frac{3}{x} = \frac{5}{6}$

(h) $\frac{3a}{b} = \frac{c}{x}$

(i) $\frac{3}{x} = \frac{5}{x} + 6$

(j) $\frac{4x - 5}{x + 2} = \frac{7}{5}$

3. Solve the inequations.

(a) $5x + 4 \geqslant 9$

(b) $7 + 3x > 1$

(c) $3 + 4x > -9$

(d) $3x + 2 < 1$

(e) $-5x \leqslant 10$

(f) $6 - 2x > 12$

(g) $-2x + 5 < -15$

(h) $8 - 3x \leqslant x - 12$

(i) $3(2x + 5) \geqslant 2(x - 4)$

(j) $\frac{1}{3}x + \frac{5}{6} < \frac{1}{2}x + 1$

4. Solve the inequations and graph the solution set on the R line.

(a) $2 + x > 5$

(b) $3(1 + x) \leqslant 6$

(c) $3x > 2x - 4$

(d) $-\frac{x}{3} \geqslant 1$

(e) $-\frac{2}{5}x < 2$

(f) $3(x - 2) - 2(x - 4) \leqslant -3$

(g) $4x + 5 \leqslant 6x - 3$

(h) $\frac{1}{4} - \frac{1}{3}x > -\frac{1}{12}$

(i) $(x + 3)^2 - (x + 2)^2 \geqslant -3$

(j) $(x - 5)(x - 4) < (x - 3)(x - 8)$

1(a) 4
(b) 6
(c) 2
(d) −3
(e) 32
(f) $\frac{21}{2}$

2(a) $\frac{q}{p}$
(c) rs
(f) $\frac{de}{3d - 1}$
(h) $\frac{bc}{3a}$

3(a) $x \geqslant 1$
(b) $x > -2$
(f) $x < -3$
(g) $x > 10$

4(a) $x > 3$
(b) $x \leqslant 1$
(c) $x > -4$
(d) $x \leqslant -3$

5. Graph the solution set.

(*a*) $x > -1$ AND $x < 3$
(*b*) $-4 \leqslant x \leqslant -2$
(*c*) $5 > x > 0$
(*d*) $x > 2$ OR $x \leqslant -4$
(*e*) $x > -3$ OR $x < 1$
(*f*) $x \leqslant -2$ AND $x > 3$
(*g*) $x \geqslant 2$ AND $x > 5$
(*h*) $x < 4$ AND $x < 0$
(*i*) $x \geqslant 3$ OR $x \geqslant 4$
(*j*) $x < 2$ OR $x < -2$
(*k*) $x \geqslant -3$ AND $x \geqslant 1$
(*l*) $-5 \leqslant x \leqslant -1$

6. Give the solution set, where
$A = \{2, 4, 6, 8, 10\}$, $B = \{5, 6, 7, 8\}$.

(*a*) $A \cap B$
(*b*) $A \cup B$

7. Find the x-intercept, the y-intercept, and then graph the relations in the xy plane.

(*a*) $2x + 5y = 20$
(*b*) $x - 3y = 6$
(*c*) $4x + y = 10$
(*d*) $7x - 2y = 28$
(*e*) $3x + 12 = 2y$
(*f*) $2y - 3x = 15$

8. Graph each relation in the xy plane, and from the graph determine the solution set of the system of equations.

(*a*) $x + 3y = 6$
$x - y = 0$

(*b*) $x - 3y = -3$
$y = 2$

(*c*) $x - y = -4$
$2x + y = 10$

(*d*) $x + y = 2$
$x = -1$

(*e*) $x + 2y = -2$
$x - 2y = 6$

(*f*) $x = 4$
$y = -2$

9. Solve the system of equations algebraically.

(*a*) $2x + y = 3$
$3x - y = 7$

(*b*) $3x + 4y = -1$
$3x - 5y = -19$

(*c*) $2x - 4y = 6$
$3x - 2y = 13$

(*d*) $5x + 3y = 9$
$3x - 2y = 13$

(*e*) $4x - 3y = 7$
$5x - 4y = 8$

(*f*) $x + 4y = 7$
$3x - 8y = 11$

(*g*) $4x - 5y = 0$
$-5x + 6y = -2$

(*h*) $\frac{1}{2}x + \frac{1}{4}y = 9$
$\frac{2}{3}x - \frac{1}{6}y = 0$

(*i*) $2x - \frac{1}{3}y = -4$
$\frac{3}{4}x - \frac{5}{6}y = 7$

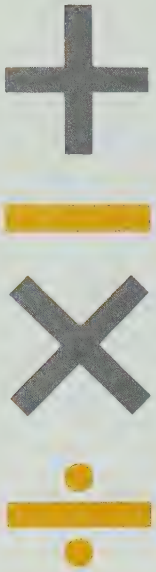
(*j*) $x + \frac{1}{2}y = -\frac{1}{2}$
$4 - 3(y - 2) = -5$

6(a) $\{6, 8\}$

7(a) x-intercept
y-intercept $= 4$
(b) x-intercept $= 6$
y-intercept $= -2$
(e) x-intercept $= -4$
y-intercept $= 6$

8(b) S.S. $= \{(3, 2)\}$
(c) S.S. $= \{(2, 6)\}$
(d) S.S. $= \{(-1, 3)\}$

9(a) $x = 2, y = -1$
(b) $x = -3, y = 2$
(c) $x = 5, y = 1$
(d) $x = 3, y = -2$
(e) $x = 4, y = 3$



10. Graph the regions given by the inequations in the xy plane.

(*a*) $x - y \geqslant 0$	(*f*) $x \leqslant -3$
(*b*) $x \geqslant 3$	(*g*) $5x \leqslant 5 + 2y$
(*c*) $x + 2y > 4$	(*h*) $y \geqslant -4$
(*d*) $y < 2$	(*i*) $2x < 3y$
(*e*) $3x + 2y < 6$	(*j*) $y > 0$

11. Graph the regions in the xy plane.

(*a*) $x \geqslant 3$ AND $y \geqslant 2$	(*h*) $x \leqslant -6$ OR $2y \leqslant 7$
(*b*) $x > 2$ OR $y < -1$	(*i*) $2x - y + 4 \geqslant 0$ AND $x + y \leqslant 1$
(*c*) $x \geqslant -1$ AND $y \leqslant 4$	(*j*) $2x - y + 6 < 0$ OR $y < x$
(*d*) $2x \leqslant 5$ OR $3y \geqslant 4$	(*k*) $3x + 2y \geqslant 6$ AND $3x + 4y \leqslant 12$
(*e*) $x < 5$ AND $y > -3$	(*l*) $x + y > -4$ OR $x - 2y + 1 > 0$
(*f*) $x > -3$ OR $y > -4$	(*m*) $x + 3y + 3 > 0$ AND $x < 3$
(*g*) $x \leqslant 4$ AND $y < 6$	(*n*) $x - 2y > 4$ OR $y > -1$

12. The sum of two numbers is 43. Five times one number added to 6 times the other is 85. What are the numbers?

13. The sum of two numbers is 62. Three times the smaller exceeds the larger by 2. Find the numbers.

14. The difference of two numbers is 27. Twice the first exceeds 7 times the second by 14. What are the numbers?

15. There are twice as many girls as boys in a mathematics class. Including the teacher there are 34 people in the room. How many boys are there, and how many girls?

16. Mary bought 6 bags of potato chips and 4 doughnuts for \$1.26. Joan paid 90 cents at the same store for 3 bags of potato chips and 5 doughnuts. Find the price of 1 bag of potato chips and of 1 doughnut.

17. From an automatic coffee maker a person can get a cup of coffee with cream and sugar for 25¢, and a cup of coffee without for 20¢. At the end of the day there was \$8.45 in the machine, and 36 cups had been used. How many cups of coffee with cream and sugar had been sold?

18. For a school fund raising campaign, Brian sold 10 boxes of cards and 2 chocolate bars, making a profit for the school of \$4.30. June sold 6 boxes of cards and 5 chocolate bars for a profit of \$3.15. What profit did the school make on one box of cards and on 1 chocolate bar?

12. 173, −130

14. 35, 8

16. 1 bag of potato chips cost 15¢.
1 doughnut cost 9¢.

17. 25

19. Simplify.

(a) $a^2 \times b^3 \times a^4 \times b^7$ (c) $5^3 \times 6^4 \times 5^6 \times 6^8$

(b) $(2x^3)(-3y^2)(-y^4)$ (d) $(4^6)(4^2)(4)$

20. Simplify.

(a) $\dfrac{7^5 \times 6^4}{6^{-3} \times 7^3}$ (c) $\dfrac{6a^{\frac{5}{2}}}{3a^{-\frac{1}{2}}}$

(b) $\dfrac{5^2}{5^{\frac{1}{4}}}$ (d) $\dfrac{15a^5b^{-2}c^3}{5a^8b^{-6}c^{-4}}$

21. Evaluate.

(a) $(-8)^{\frac{1}{3}}$ (d) $(9a^2b^3)^0$

(b) $(-1)^{-4}$ (e) $(-2x)^{-2}$

(c) $(-1)^{-3}$ (f) $(-1)^9$

22. Simplify.

(a) $t(t^2 + 3t) - 2t(t - 4t^2) - (-t^3)$

(b) $r(r^2 - 4) - r^2(5 - 3r) - 2(r^2 + r)$

(c) $3s(2 - 4s + 2s^2) - 2s(-6s + 3 + 3s^2)$

23. Factor.

(a) $4a^2b + 2ab$ (g) $50x^2 - 72y^2$

(b) $nr^2 - 3rn + 2n$ (h) $8x^3 - y^3$

(c) $k(x + y) + 3(x + y)$ (i) $27a^3 + 8b^3$

(d) $2(w + t)(w - t) - (w - t)$ (j) $k^3 - 6k^2 + 8hk - 48h$

(e) $9p^2 - 24p + 16$ (k) $a^3 - 12a^2 - 2a + 24$

(f) $12x^2 + 36x + 27$ (l) $x^3 - 4x^2 - 3x + 12$

24. Simplify.

(a) $\dfrac{5}{x+3} + \dfrac{6}{x+2}$ (c) $\dfrac{5k}{k^2 - h^2} + \dfrac{4}{k+h} - \dfrac{2}{k-h}$

(b) $\dfrac{1}{y-5} - \dfrac{2}{y+2}$ (d) $\dfrac{5}{x-y} + \dfrac{2x}{x^2 - y^2} - \dfrac{7}{x-y}$

25. Solve.

(a) $4k^2 - 9 = 0$ (d) $y^2 + 4y = 5$

(b) $p^2 + 10p + 25 = 0$ (e) $(x - 2)(x + 3) = 14$

(c) $5t + 10t^2 = 0$ (f) $(2x + 3)^2 - (x - 1)^2 = -7$

26. Expand.

(a) $(\sqrt{11} + 3)^2$ (d) $(3\sqrt{7} - 2\sqrt{5})^2$

(b) $(\sqrt{7} - 4)(\sqrt{2} - 3)$ (e) $(\sqrt{6} + 4\sqrt{2})(\sqrt{7} + 3\sqrt{5})$

(c) $(6 + \sqrt{5})(2 - \sqrt{5})$ (f) $(\sqrt{a} + \sqrt{b})^2$

19(a) a^6b^{10}
(b) $6x^3y^6$

20(c) $2a^3$
(d) $3a^{-3}b^4c^7$

21(a) -2
(b) 1
(c) -1

22(a) $10t^3 + t^2$

23(a) $2ab(2a + 1)$
(b) $n(r - 1)(r - 2)$
(c) $(k + 3)(x + y)$
(d) $(2w + 2t - 1)(w - t)$
(e) $(3p - 4)^2$

24(a) $\dfrac{11x + 28}{(x + 3)(x + 2)}$
(c) $\dfrac{7k - 6h}{k^2 - h^2}$

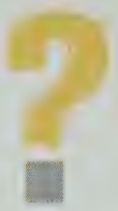
25(b) -5
(d) -5 or 1
(e) -5 or 4

26(a) $20 + 6\sqrt{11}$
(c) $7 - 4\sqrt{5}$
(d) $83 - 12\sqrt{35}$

Summary of Units 1 to 8

	Do you understand . . . ?	*Can you . . . ?*
Unit 1		
Power Terms and Their Uses		
1.1 Get-Set Exercise	integers, order of operations	add, subtract, multiply, and divide with integers
1.2 Natural Numbers as Powers	power, base, exponent the use of exponents in multiplication and division	use LEM, LED, LEP, LEPQ, and LEPP to simplify expressions
1.3 Zero and Negative Numbers as Powers	the meaning of x^0 and x^{-a}	use LEM, LED, LEP, LEPQ, and LEPP to simplify expressions with zero and negative exponents
1.4 Rational Exponents	the meaning of $x^{\frac{m}{n}}$	simplify expressions involving rational exponents
1.5 Standard Form and Its Uses	how to convert large and small numbers to standard form	use standard form to estimate answers
1.6 Further Uses of Standard Form	the use of the graph of $y = 10^x$	solve problems using the graph of $y = 10^x$
Unit 2		
Operations With Algebraic Terms		
2.1 Get-Set	the addition, subtraction, and multiplication of algebraic terms	simplify expressions by collecting like terms
2.2 Multiplying Binomials	the method of multiplying binomials	use FOIL to find a product
2.3 Division	how to divide by a monomial or by a binomial	
2.4 Squaring	that squaring is just a short way of indicating multiplication	square a monomial or a binomial
2.5 A Special Product	why FOIL yields a special product when the sum and difference of two quantities are multiplied	find the difference of squares quickly
2.6 Cube of a Binomial	the pattern that results when a binomial is cubed	cube any binomial

	Do you understand . . . ?	*Can you . . . ?*
Unit 3		
Factoring Algebraic Expressions		
3.1 Common Factoring	how to find a common factor	factor an expression by common factoring
3.2 Factoring a Difference of Squares	what a binomial looks like when it is a difference of squares	factor a difference of squares
3.3 Factoring a Perfect Square Trinomial	the special form of a perfect square trinomial	factor a perfect square trinomial quickly
3.4 Factoring by Grouping	what to look for when you group in pairs what to look for when you group in 3 and 1	identify quickly when a polynomial should be grouped 2 and 2 or 3 and 1
3.5 Further Factoring of Trinomials	the decomposition method (or do you prefer the inspection method?)	explain how you decide upon the signs to be used in the binomial factors
3.6 Sum or Difference of Cubes	what a sum or difference of cubes looks like	state the factors quickly and explain how the signs are chosen
Unit 4		
Uses of Factoring		
4.1 Multiplying and Dividing in Algebra	the use of factoring to simplify multiplication and division	multiply and divide algebraic expressions
4.2 Adding and Subtracting Rational Numbers	how to find the LCD	add or subtract accurately
4.3 Solving Quadratic Equations by Factoring	why if $ab = 0$, then $a = 0$ or $b = 0$ roots of an equation	solve a quadratic equation by factoring



	Do you understand . . . ?	*Can you . . . ?*
Unit 5		
Working With Radicals		
5.1 Radicals Exist	why we have to bother about rationals	show that radicals do exist
5.2 Operations With Radicals, Part 1	the rule for multiplication and division of radicals	simplify radical expressions involving multiplication and division
5.3 Operations With Radicals, Part 2	why like radicals are the only radicals that can be added or subtracted	change a mixed radical to an entire radical and vice versa
5.4 Rationalizing	why like radicals multiply to rationals why conjugate radicals multiply to rationals	rationalize any denominator containing radicals
Unit 6		
Working With Equations and Inequations—One Variable		
6.1 Review of Equations in One Variable	how equations can be "juggled" in order to solve for the variable	solve equations involving brackets and fractions
6.2 Solving Inequations in One Variable	why the inequality sign must be reversed when multiplying or dividing by a negative	solve any equation
6.3 Graphing the Solution Set of Equations and Inequations	the use of the real number line	graph equations and inequations on a number line
6.4 The AND, OR Connectives	the difference between AND and OR in mathematics	use the number line to solve AND, OR situations
6.5 Uses of Factoring	how factoring and the number line are combined to solve AND and OR problems	

	Do you understand . . . ?	*Can you . . . ?*
Unit 7		
Equations and Inequations — Two Variables		
7.1 Relations Revisited	the use of intercepts in graphing straight lines	draw the graph of any linear relation
7.2 Intersections and Unions — Two Variables	the significance of AND and OR when graphing two linear relations	explain why AND and OR give different results
7.3 Intersections by Algebra	the three methods for solving problems algebraically	describe the method you prefer in detail verify your solution
7.4 Inequations Involving AND and OR	how to locate the proper region once the "boundary" line has been drawn how to decide whether the boundary line is dotted or solid	remember the strategy —boundary line —determine region —is it intersection or union —solid lines where necessary
7.5 Some Non-Linear Graphs	why many more points are required for these graphs	identify the equation of a non-linear graph
Unit 8		
Problem Solving		
8.1 Translating English to Algebra	how to translate English into algebraic statements	translate English statements into equations with one variable
8.2 Problems With Two Unknowns	how to solve problems with two unknowns	solve number, weight, and money problems with two variables
8.3 More Problems With Two Unknowns	how to solve problems with two unknowns	solve age, perimeter, and mixture problems with two variables

Review Exercise, Units 1 to 8

Discussion

Can you give an example to illustrate?

LEM
LED
LEP
LEPQ
LEPP

1. State with positive exponents.

   (a) m^{-5} (d) a^{-7} (g) $\frac{1}{9^{-2}}$

   (b) t^{-2} (e) x^{-4} (h) t^{-25}

   (c) $\frac{1}{3^{-2}}$ (f) $\frac{1}{7^{-3}}$ (i) $\frac{1}{e^{-10}}$

2. Simplify.

   (a) $\frac{m^2 \times m^5}{m^3}$ (c) $\frac{a^4 \times a^5}{a^6 \times a^3}$ (e) $c^6 \times c^{-6}$

   (b) $\frac{t^7 \times t^5}{t^3}$ (d) $b^7 \div b^4$ (f) $e^2 \div e^7$

3. Express using rational exponents.

   (a) $\sqrt[5]{b}$ (c) $\sqrt[7]{m^4}$ (e) $\sqrt[7]{c^3}$

   (b) $\sqrt[3]{a^2}$ (d) $\sqrt{e^7}$ (f) $\sqrt[3]{m^2}$

4. State without powers of 10.

   (a) 3.62×10^5 (c) 5.72×10^{-3} (e) 5.19×10^2

   (b) 1.83×10^3 (d) 4.83×10^{-1} (f) 4.31×10^{-4}

5. Express in standard form.

   (a) 0.000 019 5 (c) 38.12 (e) 0.0172

   (b) 496 (d) 0.000 43 (f) 86.2

6. Identify the like terms.

   (a) $2m, 3n, 2m^2, 5n, -3n^2, -5m, 3m^2$

   (b) $2e, 5f, -2e^2, 5e, -2e^3, 15f, -3e, 5e^3$

   (c) $a, -3b, 2b^2, -5a, 6a^2, -5b^2, 7a, -14a$

7. Identify the type of factoring involved in each part.

   (a) $a^2 - b^2$ (h) $5x + 5y + x^2 + xy$

   (b) $2x + 2y + 2z$ (i) $a^2 - 6a + 9 - b^2$

   (c) $8a^3 - b^3$ (j) $3a^2 - a - 14$

   (d) $2x^2 + 11x + 5$ (k) $4e^2 + 12e + 9$

   (e) $x^2 + 6x + 9$ (l) $x^2 - 16$

   (f) $27x^3 + y^3$ (m) $3e + 6f - 9$

   (g) $5m + 5n - 15p$ (n) $8m^3 + n^3$

Written Solutions

8. Simplify.

(a) $\dfrac{3^5 \times 3^3 \times 3^{-7}}{3^4 \times 3}$

(b) $\dfrac{x^5 \times x^4 \times x^2}{x^7 \times x^3}$

(c) $\dfrac{m^{-2} \times m^{-5} \times m^3}{m^{-3} \times m^{-7}}$

(d) $\dfrac{(3r)^3 \times (3r)^2}{(3r)^4}$

(e) $\dfrac{(-5)^{11} \times (-5)^7}{(-5)^8 \times (-5)^8}$

(f) $\dfrac{(2s)^{-4} \times (2s)^{-5}}{(2s)^{-7} \times (2s)^{-3}}$

9. Simplify.

(a) $121^{\frac{1}{2}}$

(b) $8^{\frac{1}{3}}$

(c) $125^{\frac{1}{3}}$

(d) $81^{0.25}$

(e) $32^{0.2}$

(f) $169^{0.5}$

10. Expand and simplify.

(a) $4a - 2(6a - 5b) - 4(b + 2a)$

(b) $(a + 2b)^2 - (a - 3b)(2a - 5b)$

(c) $(a - 2b)(2a^2 + 5ab - 4b^2)$

(d) $(a^{\frac{1}{2}} - 2b^{\frac{1}{2}})(a^{\frac{1}{2}} - 3b^{\frac{1}{2}})$

11. Simplify.

(a) $(16^{\frac{1}{4}})(9^0)(8^{\frac{2}{3}})$

(b) $(9a^3b^{-2})^{-2}$

(c) $(a^{\frac{3}{4}}b^{-3})^8$

(d) $(16a^4b^8)^{\frac{3}{2}}$

(e) $\dfrac{6.4 \times 10^{-7} \times 8.2 \times 10^5}{4.1 \times 10^{-3}}$

(f) $\dfrac{0.000\,65 \times 0.0020}{13\,000}$

12. Simplify.

(a) $4x - 7x + x$

(b) $2ab + 3ab - 7ab + 6ab$

(c) $5z + 6z - 4z$

(d) $7x^2 - 4x - 3 + 2x - 5x^2$

(e) $cd + 5cd + 7cd$

(f) $5ef - 2ef + 4cd - 2cd$

(g) $6x^2y - 2x^2y + x^2y$

(h) $2x^2y - 4xy + 7xy - 3x^2y$

13. Multiply.

(a) $c(2c + 1)$

(b) $3x(4x - 2)$

(c) $5y(8x - 7)$

(d) $25(xy - 2)$

(e) $(p - 2q)(p - 3q)$

(f) $(2c - 5d)(c + 3d)$

(g) $(2x - y)(2x + 3y)$

(h) $(2c + 3d)(3c - d)$

(i) $(5x - 7)^2$

(j) $(2y - 3)(2y + 3)$

(k) $(7a - 2b)(a + 3b)$

(l) $(2p - 3q)^2$

(m) $(5x + 6)^2$

(n) $(7c - 2d)(7c + 2d)$

8(a) 3^{-4}
(c) m^6
(d) $3r$
(f) $2s$

9(b) 2
(e) 2

10(a) $-16a + 6b$
(b) $-a^2 + 15ab - 11b^2$

11(a) 8
(d) $64a^6b^{12}$

12(a) $-2x$
(c) $7z$
(e) $13cd$
(g) $5x^2y$

13(a) $2c^2 + c$
(c) $40xy - 35y$
(g) $4x^2 + 4xy - 3y^2$
(h) $6c^2 + 7cd - 3d^2$
(j) $4y^2 - 9$
(n) $49c^2 - 4d^2$

14. Divide.

(a) $\frac{5q}{5}$

(b) $\frac{7a - 7b}{7}$

(c) $\frac{11f - 11e}{11}$

(d) $\frac{8m - 8n}{8m}$

(e) $\frac{x^2 + 5x + 4}{x + 1}$

(f) $\frac{x^2 - 5x + 6}{x - 2}$

Factoring Strategy

1. Check for a common factor

2. Two terms —
   - Difference of squares
   - Sum or difference of cubes

3. Three terms —
   - Perfect square
   - "Trial" or decomposition

4. Four terms (group) —
   - 2 and 2 - get a common factor
   - 3 and 1 - get a difference of squares.

15. How many terms are there in each of the following products? Don't multiply.

(a) $(x + 5)(x - 3)$

(b) $(2y - 7)^2$

(c) $(x + 5)(x - 5)$

(d) $(5x - 2y)^2$

(e) $(5x - 2y)(5x + 3y)$

(f) $(x - 2y)^3$

16. What is the sign of the x term in each of the following? Don't multiply completely.

(a) $(x + 5)^2$

(b) $(x - 2)^2$

(c) $(x + 5)(x - 3)$

(d) $(5x - 7)^2$

(e) $(5x - 2)(x - 7)$

(f) $(7x - 3)(2x + 5)$

17. One factor of $x^2 - 25$ is $x - 5$. What is the other factor?

18. One factor of $x^2 + 6x + 9$ is $x + 3$. What is the other factor?

19. One factor of $x^2 + 9x + 14$ is $x + 7$. What is the other factor?

20. Expand and simplify.

(a) $(3x - 1)^3$

(b) $(5a - 2)^3$

(c) $(2x - 5)^2$

(d) $(a + 3b)^2$

(e) $(m - 3)^3$

(f) $(2m + 5n)(2m - 5n)$

(g) $(3p + q)^2$

(h) $(r - 2s)(r + 2s)$

21. Factor.

(a) $5a + 15$

(b) $9a^2 + 18a$

(c) $3x^2 + 6$

(d) $3t^2 - 2t$

(e) $3xy + 6x^2$

(f) $4c^2 - 6c$

22. Factor.

(a) $x^2 - 36$

(b) $y^2 - 16$

(c) $100 - t^2$

(d) $4x^2 - 9$

(e) $a^2 - 49$

(f) $81 - t^4$

14(b) $a - b$
(e) $x + 4$

15(a) 3
(d) 3

16(b) −
(e) −

18. $x + 3$

20(a) $27x^3 - 27x^2 + 9x - 1$
(b) $125a^3 - 150a^2 + 60a - 8$
(e) $m^3 - 9m^2 + 27m - 27$
(f) $4m^2 - 25n^2$

21(a) $5(a + 3)$
(b) $9a(a + 2)$
(c) $3(x^2 + 2)$

22(a) $(x + 6)(x - 6)$
(b) $(y + 4)(y - 4)$
(c) $(10 + t)(10 - t)$

23. Factor.
   (a) $a^2 + 4a + 4$
   (b) $x^2 + 6x + 9$
   (c) $4r^2 + 12r + 9$
   (d) $25c^2 - 10c + 1$
   (e) $m^2 + 2m + 1$
   (f) $9t^2 - 42t + 49$

24. Factor.
   (a) $x^2 + 5x + 6$
   (b) $a^2 - 2a - 15$
   (c) $c^2 + 8c + 12$
   (d) $a^2 - 4a + 3$
   (e) $x^2 - 9x + 18$
   (f) $x^2 + 3x - 28$

25. Factor.
   (a) $6a^2 - 7a + 2$
   (b) $5x^2 + 13x - 6$
   (c) $10c^2 - c - 21$
   (d) $2x^2 + 9x + 9$
   (e) $6b^2 - b - 2$
   (f) $10s^2 + 33s - 7$

26. Factor.
   (a) $px + 5p - qx - 5q$
   (b) $x^2 + xy + ax + ay$
   (c) $a^2 - ac - ab + bc$
   (d) $x^3 - 5x^2 + 3x - 15$
   (e) $7a + 7b + xa + xb$
   (f) $3x^3 - 6x^2 - 5x + 10$

27. Factor.
   (a) $125y^3 - 8z^3$
   (b) $a^3 - 343$
   (c) $8r^3 + 1$
   (d) $27x^3 - y^3$
   (e) $m^3 + 8n^3$
   (f) $8p^3 - 125$

28. Identify the type of factoring required and then factor completely.
   (a) $2a^2 + 9a + 4$
   (b) $2r^2 + 7r + 5$
   (c) $2x^2 - 3x - 14$
   (d) $7y^2 + 4y - 11$
   (e) $12a^2 - 34a + 14$
   (f) $4y^2 + 24y + 36$
   (g) $3c^2 - 4c - 4$
   (h) $24d^2 + 23d + 5$
   (i) $b^2 + 18b + 81$
   (j) $2y^2 - 14y + 24$

29. Simplify.
   (a) $(3\sqrt{2} - 4\sqrt{5})^2$
   (b) $(6\sqrt{2} - 5)(4\sqrt{3} + 7)$
   (c) $5\sqrt{72} + 2\sqrt{98} - \sqrt{50}$
   (d) $\sqrt{25a^2b^3} + 6a\sqrt{16b^3} - 7b\sqrt{4a^2b}$

30. Rationalize the denominator.
   (a) $\dfrac{2\sqrt{5}}{3\sqrt{6}}$
   (b) $\dfrac{1}{2\sqrt{11} - 3}$
   (c) $\dfrac{\sqrt{3}}{2\sqrt{5} + \sqrt{7}}$
   (d) $\dfrac{a}{\sqrt{a} - \sqrt{b}}$

23(d) $(5c - 1)^2$
(e) $(m + 1)^2$
(f) $(3t - 7)^2$

24(a) $(x + 3)(x + 2)$
(d) $(a - 3)(a - 1)$

25(a) $(3a - 2)(2a - 1)$
(b) $(5x - 2)(x + 3)$
(f) $(5s - 1)(2s + 7)$

26(b) $(x + y)(x + a)$
(e) $(a + b)(7 + x)$

27(b) $(a - 7)(a^2 + 7a + 49)$
(c) $(2r + 1)(4r^2 - 2r + 1)$

28(a) $(2a + 1)(a + 4)$
(b) $(2r + 5)(r + 1)$
(f) $4(y + 3)^2$
(g) $(3c + 2)(c - 2)$

29(a) $98 - 24\sqrt{10}$

30(a) $\dfrac{\sqrt{30}}{9}$

(b) $\dfrac{2\sqrt{11} + 3}{35}$

31. Simplify.

(*a*) $\dfrac{2}{5a^2b} + \dfrac{3}{4ab^2}$ (*c*) $\dfrac{a^2 - 8ab + 12b^2}{a^2 - 36b^2} \times \dfrac{a^2 + 6ab}{a^2 - 4b^2}$

(*b*) $1 - \dfrac{3b - a}{a - 2b}$ (*d*) $\dfrac{5a}{a^2 + 7ab + 10b^2} - \dfrac{2}{a + 5b} - \dfrac{3}{a + 2b}$

32. Solve for x and y.

(*a*) $\dfrac{5}{4x} = \dfrac{2}{3}$ (*d*) $y^2 - 5y = 0$

(*b*) $x^2 - 7x - 8 = 0$ (*e*) $3x + p = x(2 + q) + 6$

(*c*) $\dfrac{x - 1}{3} - \dfrac{3x + 2}{4} = 5$ (*f*) $5x - 2y = 9$
$3x + 7y = -11$

33. Graph on the R line.

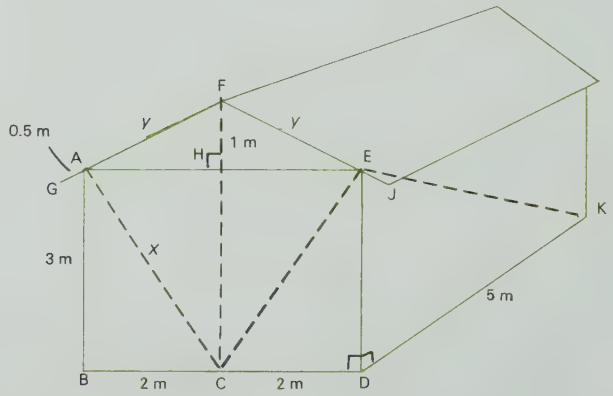
(*a*) $x \geqslant 3$ or $x \leqslant -1$ (*b*) $5 \leqslant x \leqslant 8$

34. Find the x- and y-intercepts and graph on the xy plane.

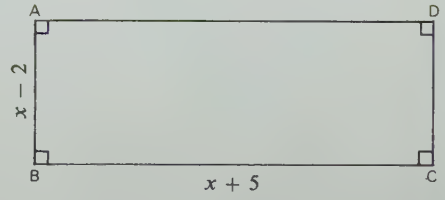
(*a*) $6x - 4y = 12$ (*b*) $4x + 2y = 20$

35. From the diagram, find:

(*a*) $x = \text{AC}$
(*b*) $y = \text{FE} = \text{FA}$
(*c*) $\text{AC} + \text{CE} + \text{AF} + \text{FE}$
(*d*) KE
(*e*) $\dfrac{\text{KE}}{\text{FE}}$ (rationalize the denominator)
(*f*) $\dfrac{\text{KE}}{\text{FJ}}$ (rationalize the denominator)

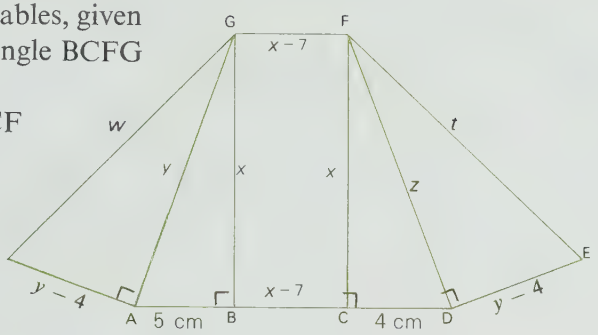


36. The area of rectangle ABCD is 44 m^2. Find x.



31.(a) $\dfrac{8b + 15a}{20a^2b^2}$
(c) $\dfrac{a}{a + 2b}$
32.(a) $x = \dfrac{15}{8}$
(d) $y = 0$ or 5
34.(a) x-intercept is 2
y-intercept is -3
35.(a) $\sqrt{13}$
(b) $\sqrt{5}$
(c) $2\sqrt{13} + 2\sqrt{5}$

37. From the diagram, find the value of the variables, given the area of rectangle BCFG is 60 cm^2.
(*a*) x = BG, CF
(*b*) y = AG
(*c*) z = FD
(*d*) t = FE
(*e*) w = HG



(*f*) Use a slide rule or calculator to find out which is larger, w or t, and by how much.

38. On a large diagram draw the boat described by these regions.
(*a*) $x \geqslant 0$ AND $y \geqslant 1$ AND $7x + 4y \leqslant 28$
(*b*) $x \leqslant 0$ AND $y \geqslant 1$ AND $2x - 3y + 24 \geqslant 0$
(*c*) $y \leqslant 0$ AND $x + y \leqslant -12$ AND $y \geqslant -2$ AND $x - 2y \leqslant 8$

39. Three boys go for a 1.5 h ride on their snowmobiles. The average velocities of the vehicles are $9\sqrt{12}$, $4\sqrt{48}$, and $6\sqrt{27}$ km/h.
(*a*) Find the distance each boy travelled. Distance equals velocity $\times$ time.
(*b*) What is the total distance travelled by the three boys?
(*c*) If gasoline costs 3¢/km, how much did the outing cost them for gas?

40. Wade Tracker had 16 ℓ of gas and 0.5 ℓ of oil put into his snowmobile. It cost him \$4.46. Rick Rackitup left his snowmobile at the same service station and paid \$6.10 for 20 ℓ of gas and 1 ℓ of oil. What was the price of a litre of oil?

41. The velocity of a wave = frequency $\times$ wavelength. Light and radio waves travel with a speed of 3.0×10^{10} cm/s. Find
(*a*) the wavelength of yellow light whose frequency is 5.0×10^{14} Hz.
(*b*) the wavelength of a radio station's signal whose frequency is 820×10^{3} Hz.
(*c*) the frequency of a TV signal whose wavelength is 400 cm.

37.(a) 12
(b) 13
39.(a) $27\sqrt{3}$ km, $24\sqrt{3}$ km, $27\sqrt{3}$ km
41.(a) 0.6×10^{-4} cm
(c) 7.5×10^{7} Hz

Unit 9
Lines and Their Equations

9.1 Get-Set Exercise

1. Express each ratio in lowest terms.
   (*a*) 9:12 (*c*) 45:15 (*e*) 51:17
   (*b*) $\frac{36}{4}$ (*d*) $\frac{19}{57}$ (*f*) 18:72

2. Determine which of the following pairs of ratios are equal.
   (*a*) 3:15, 12:45 (*e*) 18:16, 27:24
   (*b*) 7:5, 35:49 (*f*) 27:33, 36:44
   (*c*) 6:12, 18:36 (*g*) 4:5, 24:25
   (*d*) 4:6, 9:9 (*h*) 8:3, $4:\frac{3}{2}$

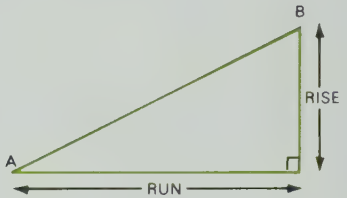
3. State the value of x that makes each of the following sentences true, $x \in W$.
   (*a*) $x:9 = 6:9$ (*e*) $\frac{1}{6} = \frac{3}{x}$
   (*b*) $14:x = 14:20$ (*f*) $\frac{7}{3} = \frac{x}{12}$
   (*c*) $20:25 = x:25$ (*g*) $9:36 = x:24$
   (*d*) $5:\frac{7}{2} = 5:x$ (*h*) $7:x = 42:54$

4. Solve the following.
   (*a*) $a:9 = 2:3$ (*d*) $9:l = 3:4$
   (*b*) $4:b = 12:15$ (*e*) $5:9 = 10:f$
   (*c*) $20:25 = d:5$ (*f*) $8:6 = h:3$

9.2 Slopes

A The cross sections of three ski hills are shown. Which is the steepest? Which has the least slope? In each case a skier travelling from the top to the bottom of the hill covers the same horizontal distance but the vertical distance is different.

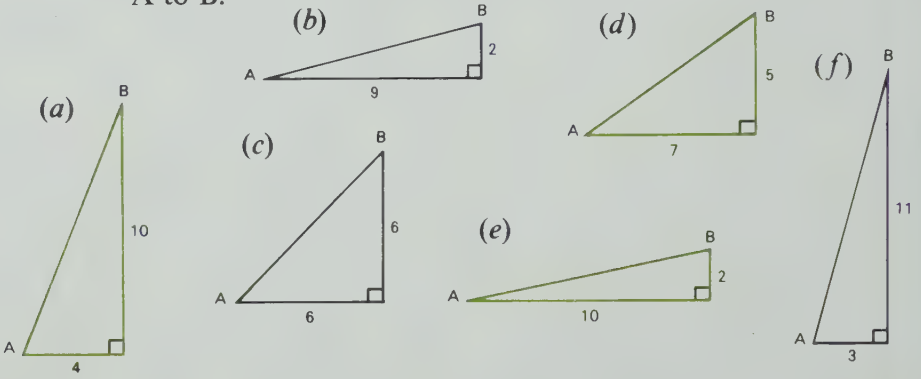
Slope is a measure of steepness, and for a hill the measure of slope depends both on vertical height and horizontal distance.



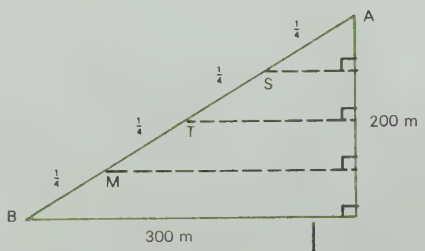
Slope up the hill is the same as slope down the hill.

To get back up a hill, A to B, we travel
(*a*) a horizontal distance called the *run*, and
(*b*) a vertical distance called the *rise*.

1. For each of the following hills state the rise and the run from A to B.

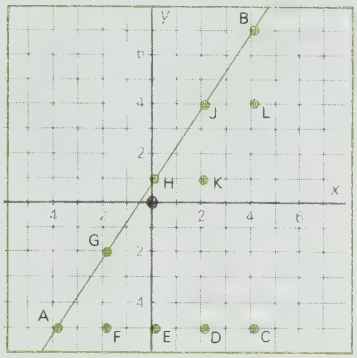


B Sam Sitzmark and Chris Christie are coming down Devil's Run.



(*a*) They spill at S, a quarter of the way down the hill. How far have they come vertically? How far horizontally?
(*b*) What is the rise from S to A? The run from S to A?
(*c*) Sam and Chris turn and stop at T. What is the rise from here to the top? What is the run?
(*d*) They stop abruptly to meet some skiers at M, three-quarters of the way down the hill. What distance have they travelled horizontally? How far vertically?

C Let's look at Superskier Slide using the coordinate axes.



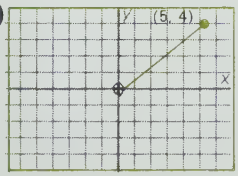
1. Copy and complete the table.

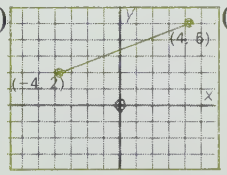
rise	*run*	$\frac{\textit{rise}}{\textit{run}}$
CB	AC	$\frac{CB}{AC} =$
DJ	AD	$\frac{DJ}{AD} =$
KJ	HK	$\frac{KJ}{HK} =$
EH	AE	$\frac{EH}{AE} =$
LB	JL	$\frac{LB}{JL} =$

Our table suggests that, for any two points on the line AB, the fraction $\frac{\text{rise}}{\text{run}}$ always has the same value. This value is called the *slope of the line.*

$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$

2. Find the slope of each of the following segments.

(a)


(c)


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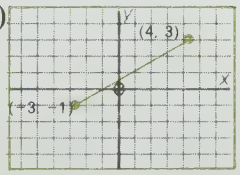

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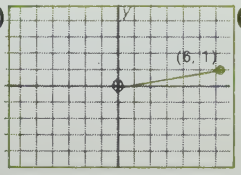

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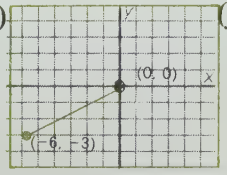

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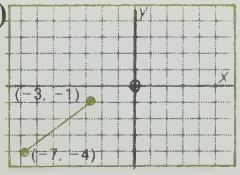
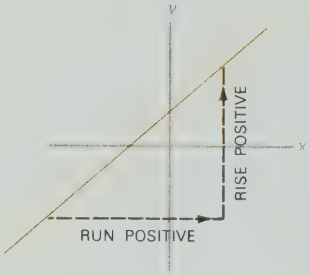


SLOPES

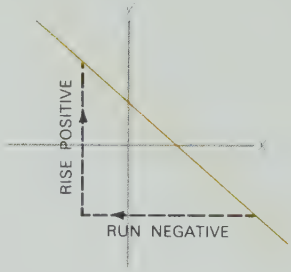
POSITIVE

NEGATIVE

D When we use the coordinate axes, we must be able to distinguish between two kinds of slopes.



Slope positive

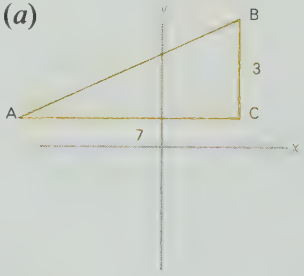


Slope negative

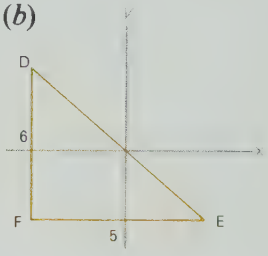
Example

State the rise and run in the following cases, and determine the slope.

(*a*)



(*b*)



Solution

(*a*) rise CB $= 3$
run AC $= 7$

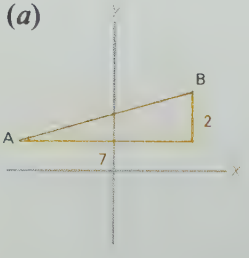
$\therefore$ slope of AB $= \frac{3}{7}$

(*b*) rise FD $= 6$
run EF $= -5$

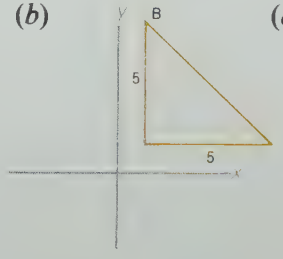
$\therefore$ slope of ED $= \frac{6}{-5} = -\frac{6}{5}$

1. (*a*) State the rise and run from A to B in each of the following. Use a sign to indicate the direction.
   (*b*) Calculate the slope in each case.

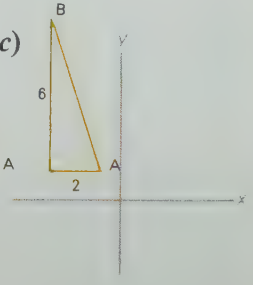
(*a*)



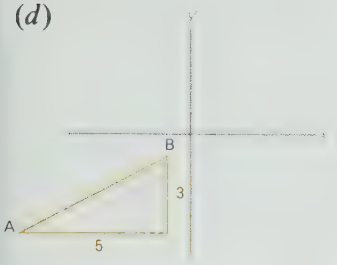
(*b*)



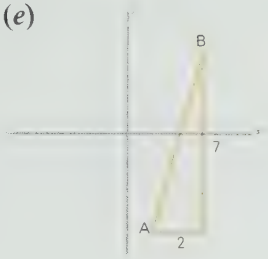
(*c*)



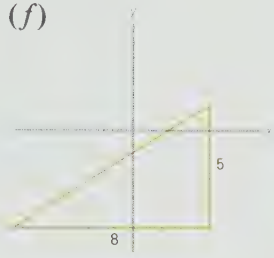
(*d*)



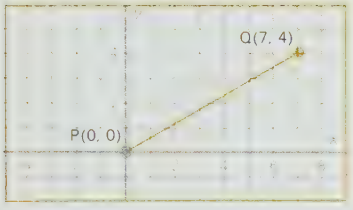
(*e*)

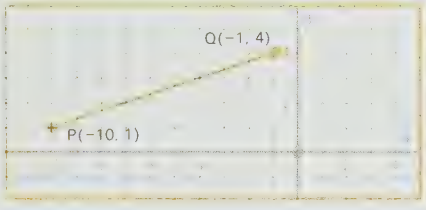


(*f*)



E We can find the rise and run from P to Q if the coordinates are known.





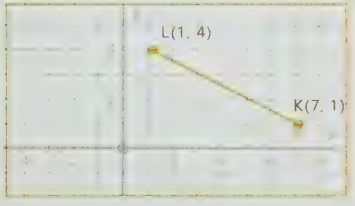
For each segment, find
(*a*) (*y* coordinate of Q) − (*y* coordinate of P),
(*b*) (*x* coordinate of Q) − (*x* coordinate of P),
(*c*) rise of PQ,
(*d*) run of PQ.

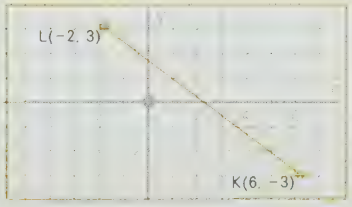
Now compare the values.What conclusion would you make?

Rise from P to Q = (*y* coordinate of Q) − (*y* coordinate of P)
Run from P to Q = (*x* coordinate of Q) − (*x* coordinate of P)

1. Find the values of the rise and run from K to L in each of the following.

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x}$$



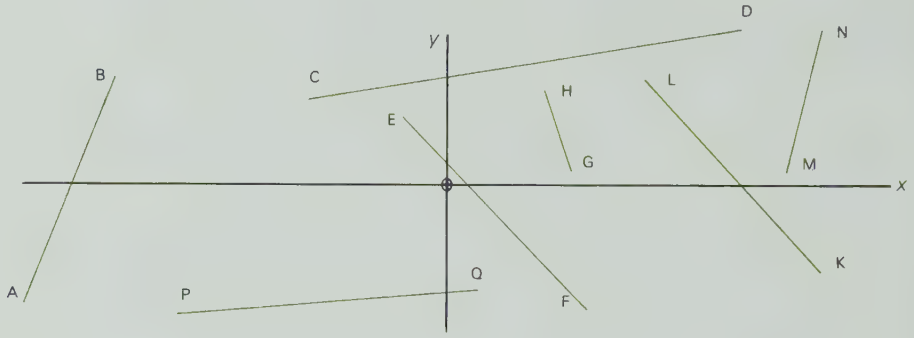


2. Now calculate the rise and run by using the rule you developed. Compare the results.

Exercise 9.2

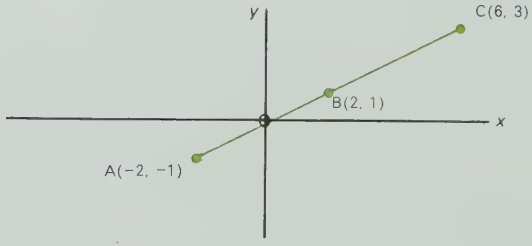
Discussion

1. In the accompanying diagram identify the lines that have a positive slope and those that have a negative slope.



2. (*a*) In the accompanying diagram determine the slopes of the line segments AB, AC, and BC.
   (*b*) What property of the line does this illustrate?
   (*c*) Determine the rise and run of line segment CA. What is the slope of line segment CA?

Slope = $\frac{\text{change in } y}{\text{change in } x}$



Written Solutions

3. Find the slope of the line segments joining the following pairs of points.

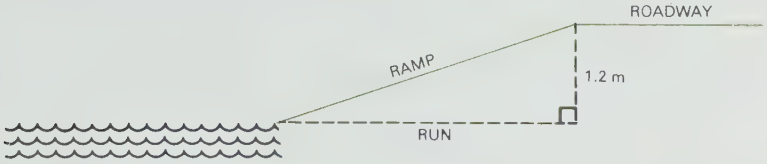
   (*a*) (0, 0) and (4, 4)
   (*b*) (−6, −2) and (4, 3)
   (*c*) (−3, −1) and (−5, 7)
   (*d*) (4, −1) and (−2, 2)
   (*e*) (0, 0) and (−3, −3)
   (*f*) $\left(\frac{2}{5}, \frac{-1}{3}\right)$ and $\left(\frac{3}{5}, \frac{2}{3}\right)$
   (*g*) $\left(\frac{1}{4}, \frac{1}{6}\right)$ and $\left(\frac{3}{4}, \frac{-1}{2}\right)$
   (*h*) (−2, 4) and (3, 4)
   (*i*) (4, 1) and (−7, 1)
   (*j*) (3, 2) and (3, −7)

The last three are interesting!

3.(a) 1 (f) 5
(c) −4 (h) 0
(e) 1 (i) 0

4. What is the slope of a line that is
   (*a*) parallel to the x axis?
   (*b*) parallel to the y axis?

5. A boat launching ramp has a slope of $\frac{2}{15}$. If the top of the ramp is 1.2 m above water level, what is the horizontal distance from the top of the ramp to the water?



6. An approach to a highway bridge passing over a thruway has a slope of $\frac{1}{50}$. What is the rise over a run of 2000 m?

7. Copy and complete the following table.

Rise	*Run*	*Slope*
3	10	
	5	$\frac{2}{5}$
6		$\frac{1}{5}$
	8	$\frac{1}{16}$
-9		3
7		14
	-3	$\frac{3}{7}$
8	4	
$\frac{1}{3}$		$-\frac{2}{7}$

8. A line that passes through the point P$(-2, 5)$ and Q$(4, y)$ has a slope of $\frac{1}{2}$. Find the value of y.

9. Find the point P on the y axis such that the line segment from this point to Q$(4, 7)$ has a slope of $\frac{5}{2}$.

Slope = $\frac{\text{rise}}{\text{run}}$

Rise = change in y
Run = change in x

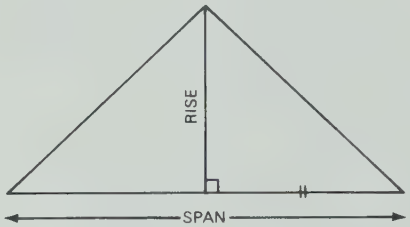
4.(a) 0
5. 9 m
8. 8
9. (0, −3)

10. The slope of the line segment joining P$(-3, 2)$ to Q$(x, 4)$ is $\frac{1}{3}$. Determine the value of x.

11. The slope of the line segment joining $(0, 4)$ to $(-5, -5)$ is equal to the slope of the line joining $(6, 4)$ to $(x, -5)$. Determine the value of x.

12. Express the slope of the line segment joining P(x_1, y_1) and Q(x_2, y_2) in terms of the coordinates of P and Q.

13. The pitch of a roof is found by the fraction $\frac{\text{rise}}{\text{span}}$ where the span is the distance indicated in the diagram.



Pitch $= \frac{\text{rise}}{\text{span}}$

How would you find the slope of the roof?

Copy the table and fill in the missing information.

Rise	*Span*	*Pitch*	*Slope*
4	18		
2		$\frac{1}{5}$	
	22	$\frac{2}{11}$	
5		$\frac{1}{6}$	
	27	$\frac{2}{9}$	
10		3	
	8	$\frac{1}{3}$	
	14	2	

10. 3
11. 1
12. $\frac{y_2 - y_1}{x_2 - x_1}$

9.3 Equations of a Straight Line

A Recall that linear equations of the form $y = x + 7$, $3x + 2y = 5$, $x = 5$, $y = 7$, $3x - 4y = 9$ are all equations of straight lines and all have straight line graphs.

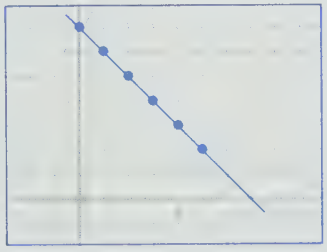
How can we tell that $x + y = 7$ represents a straight line? What do x and y represent in the equation?

Find some numbers that make $x + y = 7$ true. Choose some values of x such as 0, 1, 2, 3, 4, 5 and find corresponding values of y.

$x + y = 7$

x	y	(x, y)
0	7	(0, 7)
1	6	(1, 6)
2	5	(2, 5)
3	4	(3, 4)
4	3	(4, 3)
5	2	(5, 2)

Since (0, 7), (1, 6), (2, 5), (3, 4), (4, 3) and (5, 2) all satisfy $x + y = 7$, we can say that these points lie on the graph of $x + y = 7$. But we can get as many points as we like by choosing other values of x. For example, if $x = \frac{1}{2}$, then $y = 6\frac{1}{2}$. And if $x = 0.3$, then $y = 6.7$, and so on.



To represent all the points that are on the graph of $x + y = 7$, we join the plotted points with a solid line.

Why can we join these points with a solid line?

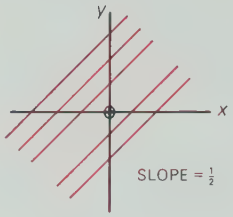
Therefore, the x and y in the equation $x + y = 7$ are coordinates (x, y) of a point on the line. In fact, the (x, y) can represent any point on the line.

(*a*) Find values of x and y that make $y = 3x - 2$ true and use these values to find points on the line.

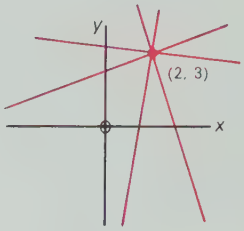
(*b*) Find values of x and y that make $y = 7$ true and use these values to find points on the line.

$y = 7$ may be written $0x + 1y = 7$

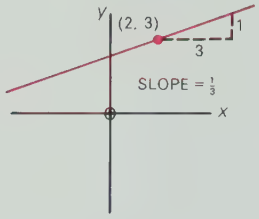
B What do you need to know in order to locate a line or graph? Is the direction of the line enough? Is one point on the line enough? Let's find out.



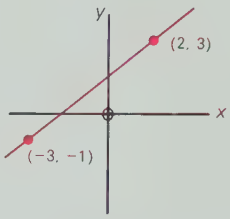
Direction alone cannot locate a line.



One point on the line is not enough to locate the line.



One point and a direction fix a line.



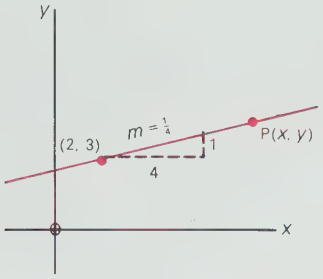
Two points fix a line.

1. On a Cartesian graph locate the lines that pass through
   (*a*) (0, 0) and (4, 3)
   (*b*) (2, 3) and (−5, 1)
   (*c*) (−2, −1) and (1, 9)
2. On a Cartesian graph locate the following lines.
   (*a*) passes through (0, 0) with a slope of $\frac{1}{7}$
   (*b*) passes through (1, 1) with a slope of 3
   (*c*) passes through (−2, −5) with a slope of $\frac{2}{5}$
   (*d*) passes through (2, 1) with a slope of $-\frac{1}{3}$
   (*e*) passes through (−2, 4) with a slope of −3

C To find the equation of a line, given a point and the slope, we need to find an equation in x and y from this given information. Remember that x and y are the coordinates of any point on the line.

Example

Find an equation of the line through (2, 3) with a slope of $\frac{1}{4}$.



Solution

Let P(x, y) represent any point on the line. Since the slope between any two points on the line is $\frac{1}{4}$ the slope of the line segment joining (x, y) to (2, 3) is $\frac{1}{4}$.
Therefore we can write

$$\frac{y-3}{x-2} = \frac{1}{4}$$

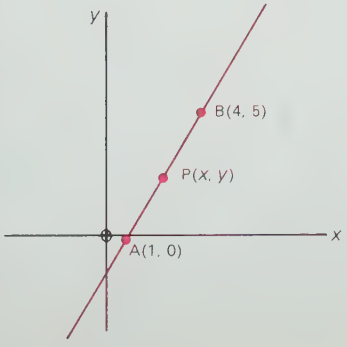
By cross-multiplying,

$$4(y-3) = 1(x-2)$$
$$\therefore 4y - 12 = x - 2$$
$$\therefore x - 4y + 10 = 0 \text{ is an equation of the required line.}$$

D We can also find the equation of a line, given two points on the line.

Example

Find an equation of the line through A(1, 0) and B(4, 5).



$$\text{Slope of AB} = \frac{5-0}{4-1}$$
$$= \frac{5}{3}$$

Thus if (x, y) represents any other point on the line we can write

$$\frac{y-0}{x-1} = \frac{5}{3}$$
$$\therefore 3y = 5x - 5$$
$$\therefore 5x - 3y - 5 = 0 \text{ is an equation of the required line.}$$

We can represent the slope with the letter m.
$m = \frac{1}{4}$

Slope = $\frac{\text{change in } y}{\text{change in } x}$

Slope between (x, y) and $(2, 3)$ is $\frac{y-3}{x-2}$

Slope = $\frac{y_2 - y_1}{x_2 - x_1}$

Success again!
We're getting good at this!

Exercise 9.3

Discussion

1. (*a*) Name two points on the x axis.
   (*b*) What is the slope of the x axis?
   (*c*) What is an equation of the x axis?

2. (*a*) Name two points on the y axis.
   (*b*) What is the slope of the y axis?
   (*c*) What is an equation of the y axis?

Written Solutions

3. Determine the slope of the line passing through the following pairs of points.

   (*a*) (2, 1), (7, 3)
   (*b*) (−2, −4), (−4, 5)
   (*c*) (−2, 3), (3, −1)
   (*d*) (3, 4), (−2, −1)
   (*e*) (10, 5), (−16, 13)
   (*f*) (2, −3), (−3, −9)
   (*g*) $\left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{3}{2}, \frac{1}{4}\right)$
   (*h*) $\left(\frac{-2}{3}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{5}{6}\right)$

4. In each of the following find an equation of the line through the point and with the slope given.

   (*a*) A(0, 0), slope $\frac{1}{2}$
   (*b*) B(2, 5), slope −3
   (*c*) C(−8, −2), slope $\frac{-4}{5}$
   (*d*) D(0, −3), slope 2
   (*e*) E(5, 0), slope $\frac{2}{3}$
   (*f*) F(−2, 3), slope 4
   (*g*) G(0, 2), slope 3
   (*h*) H(4, 1), slope $\frac{-1}{2}$
   (*i*) I(−2, 2) slope 0
   (*j*) J$\left(\frac{-2}{3}, \frac{1}{4}\right)$, slope −2

5. Find an equation of a line that passes through the origin with slope given in each of the following.

   (*a*) −1 (*b*) 3 (*c*) m (*d*) 0

6. (*a*) Find the slope of a line passing through D(2, 4) and E(−4, 4).
   (*b*) Describe the line through these points.
   (*c*) What is the equation of the line through these points?

P(x, y) represents any point on the line

3.(a) $\frac{2}{5}$ **(e)** $-\frac{4}{13}$
(c) $-\frac{4}{5}$ **(g)** $-\frac{1}{2}$

4.(a) $x - 2y = 0$
(d) $2x - y - 3 = 0$
(h) $x + 2y - 6 = 0$

5.(a) $x + y = 0$
(d) $y = 0$

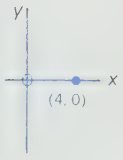
6.(a) 0

7. Determine an equation of the line through the given two points.
   (*a*) (2, 6) and (3, 7)
   (*b*) (−2, 5) and (2, −1)
   (*c*) (−3, −1) and (4, 1)
   (*d*) $\left(\frac{1}{2}, \frac{-2}{3}\right)$ and $\left(\frac{3}{2}, \frac{7}{3}\right)$
   (*e*) (−1, 4) and (−2, 2)
   (*f*) (x_1, y_1) and (x_2, y_2)

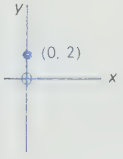
8. Find an equation of the line given the point (4, 5) and the following conditions.
   (*a*) through (0, 0)
   (*b*) with a slope of 2
   (*c*) parallel to the y axis
   (*d*) through (−4, −1)
   (*e*) parallel to the x axis

9. Determine an equation of each of the lines as described.
   (*a*) slope $\frac{1}{2}$ and passing through (0, 2)
   (*b*) slope −5 and passing through (0, −2)
   (*c*) slope m and passing through (0, b)

10. (*a*) Determine an equation of the line that has an x-intercept of 4 and a y-intercept of 2.
    (*b*) Find an equation of the line through (a, 0) and (0, b).



An x-intercept of 4 means that the line cuts the x axis at (4,0)



What does a y-intercept of 2 mean?

9.4 Equations of a Line in the Form $y = mx + b$

A Any linear equation can be rewritten in the form $y = mx + b$.

For example,

$$\text{if } 2x + y = 7$$
$$\text{then } y = -2x + 7$$

is another equation of the same line expressed in the form

$$y = mx + b$$

where $m = -2$ and $b = 7$.

But what about $x = a$?

7.(b) $3x + 2y - 4 = 0$
(d) $18x - 6y - 13 = 0$
(e) $2x - y + 6 = 0$

8.(a) $5x - 4y = 0$
(c) $x = 4$
(e) $y = 5$

9.(a) $x - 2y + 4 = 0$

10.(a) $x + 2y - 4 = 0$

1. Rewrite the following equations in the form $y = mx + b$.
   (*a*) $x + y - 2 = 0$ (*c*) $3x - y = 5$
   (*b*) $3x + y + 9 = 0$ (*d*) $2x + 3y - 6 = 0$

2. State the values of m and b in each of the equations in Question 1.

B Sometimes it is very useful to write an equation in the form $y = mx + b$ and then sketch the graph.

(*a*) $2x - y + 5 = 0$ (*b*) $2x - y = 0$ (*c*) $2x - y - 5 = 0$

(*a*) $y = 2x + 5$

x	y
0	5
1	7
2	9

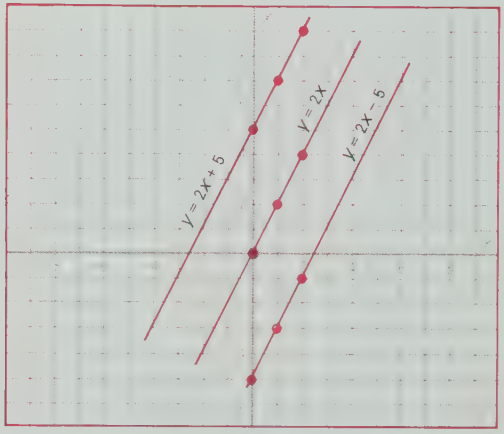
(*b*) $y = 2x$

x	y
0	0
1	2
2	4

(*c*) $y = 2x - 5$

x	y
0	−5
1	−3
2	−1

Find the y-intercept →
Find one point →
Find a check point →



Look ! They all have the same slope !

Slope of $y = 2x + 5$ is $\dfrac{7-5}{1-0} = 2$

Slope of $y = 2x$ is $\dfrac{2-0}{1-0} = 2$

Slope of $y = 2x - 5$ is $\dfrac{-3-(-5)}{1-0} = \dfrac{-3+5}{1} = 2$

Note this!

Compare the slope with the coefficient of x. What conclusion might you draw?

1. State the slope of each of the following lines and check by graphing each line.

(*a*) $y = 3x + 2$ (*c*) $y = \dfrac{-3x}{4} + 6$

(*b*) $y = \dfrac{x}{2} - 7$ (*d*) $y = 7x - 2$

2. Express each equation in the form $y = mx + b$ and state the slope of each line.

(*a*) $3x + y = 1$ (*d*) $4x + 5y = 7$

(*b*) $3x + 2y + 7 = 0$ (*e*) $x - 3y + 6 = 0$

(*c*) $x - y - 6 = 0$ (*f*) $5x + 2y = 9$

C There is more information we can get from an equation in the form $y = mx + b$. Graph the following linear equations and you will see this.

(*a*) $y = 3x + 1$ (*b*) $y = 3x - 4$ (*c*) $y = 3x + 2$

(*a*)

x	y
0	1
1	4
2	7

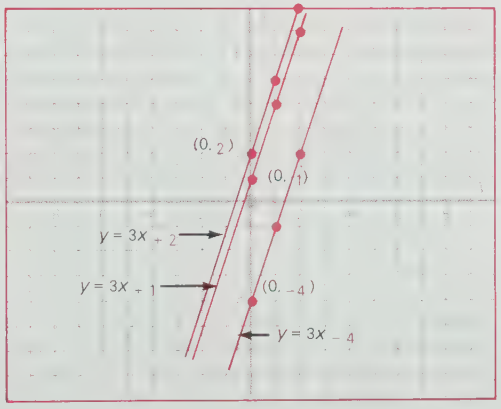
(*b*)

x	y
0	−4
1	−1
2	2

(*c*)

x	y
0	2
1	5
2	8

Find the y-intercept.
Find one point.
Find a check point.



What is the *y*-intercept of each line?
Compare this value with the value *b*.
What conclusion would you draw?

And note this!

1. What is the y-intercept of each of the following lines?

   (a) $y = \frac{x}{2} + 7$ (d) $y = x + \frac{3}{4}$

   (b) $y = 5x - 2$ (e) $y = 2x - 9$

   (c) $y = \frac{3x}{4} - \frac{1}{2}$ (f) $y = -3x + 12$

2. Rewrite each equation in the form $y = mx + b$ and name the y-intercept in each case.

   (a) $5x + 2y = 6$ (d) $x + 3y - 6 = 0$

   (b) $3x - y = 7$ (e) $4x + 7y = 8$

   (c) $x + y - 4 = 0$ (f) $x - 2y + 6 = 0$

Exercise 9.4

$y = mx + b$

SLOPE y-INTERCEPT

Discussion

1. State the slope and y-intercept of each of the following lines.

   (a) $y = -2x + 7$ (f) $y = \frac{x}{3} + 7$

   (b) $y = 3x + \frac{1}{2}$ (g) $y = \frac{-x}{2} + 3$

   (c) $y = -5x - 3$ (h) $y = 7x - \frac{1}{2}$

   (d) $y = 3x - 2$ (i) $y = 7x + 8$

   (e) $y = \frac{-1}{2}x + 2$ (j) $y = \frac{x}{4} + 7$

Written Solutions

2. By changing the following equations to the form $y = mx + b$ determine the slope and y-intercept.

   (a) $2x - y = 7$ (d) $x + y = 11$

   (b) $3x + 2y = 5$ (e) $4x + 2y = 3$

   (c) $5x - y = 4$ (f) $2x - 3y = 6$

3. Sketch graphs of each of the following lines.

   (a) A(0, 4), slope 2 (d) D(0, 3), slope -2

   (b) B(0,–2), slope $\frac{1}{2}$ (e) E(0, 5), slope $\frac{-3}{4}$

   (c) C$\left(0, \frac{-1}{2}\right)$, slope 5 (f) F(0, 1), slope -7

4. Find the equation of each line in Question 3.

5. Sketch the graphs of each of the following lines by finding a point on the line and the slope.

(a) $y = 2x - 2$ (d) $3y - 7 = 0$

(b) $y = 3x$ (e) $2x + 5y = 2$

(c) $y = \frac{1}{2}x + 7$ (f) $3x - y = 7$

6. Write an equation for each of the following lines.

(a) slope $\frac{-1}{2}$ and y-intercept 5

(b) slope 2 and y-intercept $\frac{-3}{4}$

(c) slope 5 and y-intercept 6

(d) slope $-\frac{3}{4}$ and y-intercept -7

7. Write an equation of each line given the following information.

(a) a slope of 2 (e) y-intercept 0

(b) a slope of $\frac{-2}{3}$ (f) parallel to the x axis

(c) a slope of 0 (g) passing through the origin

(d) y-intercept 5

8. Copy and complete the following table in your book.

Equation	$y = mx + b$	*Slope*	*y-intercept*	*Coordinates of one point on line*
$y = \frac{x}{5} + 2$				
$y = -3x + 7$				
$3x + 2y = 5$				
$4x + 3y = 12$				
$y = 7$				
$y = 2x$				

5(a)
Let $x = 1$
$\therefore y = 2(1) - 2$
$= 2 - 2$
$= 0$
$\therefore$ a point on the line is $(1, 0)$.

4.(a) $y = 2x + 4$
(d) $y = -2x + 3$

6.(a) $y = -\frac{1}{2}x + 5$
(c) $y = 5x + 6$

7.(a) $y = 2x + b$
(c) $y = b$
(f) $y = b$

10

Unit 10

Linear Equations and Their Applications

10.1 Direct Variation

A A pilot flew a glider plane a distance of 150 km in 60 min. During this time he descended from 3900 m to 2400 m. How much longer could he have remained in the air? How far would he have glided?

Linear equations are useful in mathematics and science because they can help solve interesting problems such as this one.

When any two variables such as x and y are related so that for all values of x and y the value $\frac{y}{x}$ remains constant, we say that y varies directly as x.

Is the reverse true?

For direct variation,
$\frac{y}{x}$ = "a constant"

1. A car travels a certain distance in a certain time according to the following chart.

Distance Travelled (km)	*Time* (h)
d	t
50	$\frac{1}{4}$
100	$\frac{1}{2}$
200	1
400	2
800	4
1000	5

In each case, $\frac{d}{t} = \frac{50}{\frac{1}{4}} = \frac{100}{\frac{1}{2}} = \frac{200}{1} = \frac{400}{2} = \frac{800}{4} = \frac{1000}{5}$

$= 200$ (a constant)

Thus d varies directly as t.

2. If a roll of carpet is 2 m wide, the area will depend on the length. Here are some possible areas of the carpet.

Length (m)	*Area* (m^2)
1	2
2	4
3	6
4	8
5	10

In each case $\frac{A}{l} = 2$ (a constant) and A varies directly as l.

$\frac{\text{area}}{\text{length}} = k$

3. From each of the following tables, calculate the value of the constant.

(*a*)

No. of people (N)	2	3	4	5	8	9	12
Cost \$(C)	3.00	4.50	6.00	7.50	12.00	13.50	18.00

(*b*)

Amount owing \$(A)	100	150	300	400	450	600
Interest \$(I)	5.00	7.50	15.00	20.00	22.50	30.00

(*c*)

Voltage (V)	8	32	64	76	88	100
Current (I)	2	8	16	19	22	25

$$\frac{N}{C} = k$$

$$\frac{A}{I} = k$$

$$\frac{V}{I} = k$$

If two variables are so related that any value of one divided by the corresponding value of the other produces a constant, then one *varies directly* as the other.

$$y \propto x$$

is the same as

$$\frac{y}{x} = k$$

4. Copy and complete the following tables if the constant in each case is as indicated.

(*a*) $k = 2$, $y \propto x$

y					20		28		
x	2	4	6	8		12		16	18

(*b*) $k = 7$, $A \propto w$

Area	14	21	35	49	84		140	
Width						15		56

(*c*) $k = \frac{1}{2}$, $d \propto t$

d				12		24.5		
t	8	9	18		35		92	422

To solve $\frac{x}{3} = 7$, multiply by 3.

$\frac{3x}{3} = 3 \times 7$

$\therefore \quad x = 21$

$y = mx + b$

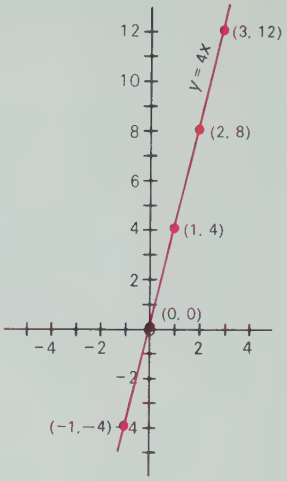
SLOPE y-INTERCEPT

If $\frac{y}{x} = k$ then $y = kx$. Compare this to the form $y = mx + b$.

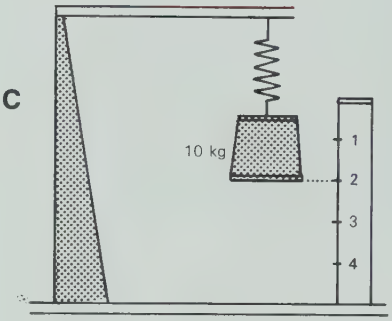
Why can't we extend the graph for negative values of y and x?

B If two variables x and y are related so that $\frac{y}{x} = 4$, we can rewrite this in the form $y = 4x$. Since this equation is of the form $y = mx + b$, its graph will be a line with slope 4 and y-intercept of 0.

y	-4	0	4	8	12
x	-1	0	1	2	3



C



1. A 10 kg mass on a spring produces a stretch of 2 cm and the stretch varies directly with the mass.
   (*a*) Find the equation representing this relation.
   (*b*) Sketch the graph.

Let y represent the stretch and x the mass.

(*a*)

$$\frac{y}{x} = k$$

If $y = 2$ and $x = 10$

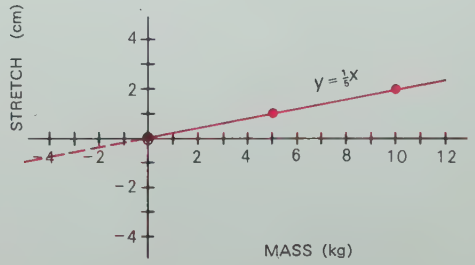
$$\frac{2}{10} = k$$

$$\therefore k = \frac{1}{5} \text{ and } \frac{y}{x} = \frac{1}{5}$$

$$\therefore y = \frac{1}{5}x$$

(*b*)

x	y
0	0
5	1
10	2
15	
20	
25	



Exercise 10.1

Written Solutions

1. Copy and complete each of the following table of values and calculate the value of the constant.

If $V \propto I$
then $\frac{V}{I} = k$
and $V = kI$

(*a*) $V \propto I$

Voltage (V)	8	12		16	24	32
Current (I)	6	9	10	12		

(*b*) $A \propto w$

Area	8	56			64	808	1632
Width	1	7	10	15			

(*c*) $c \propto d$

Circumference	6.2	15.5	27.9	31.0		55.8
Diameter	2	5	9	10	12	

Remember the name of this constant?

2. Copy and complete each of the following table of values if the constants are as given.

(*a*) $k = \frac{3}{2}$, $C \propto n$

Cost					24	33	66	
No. of guests	2	8	9	10			82	100

(*b*) $k = 4$, $d \propto t$

Distance					64	128
Time	1	2	4	8		

(*c*) $k = 5\%$, $I \propto A$

Interest				30	75	150
Amount owed	100	150	300			

1.(a)

Voltage	8	12	$\frac{40}{3}$	16	24	32
Current	6	9	10	12	18	24

$k = \frac{4}{3}$

2.(b)

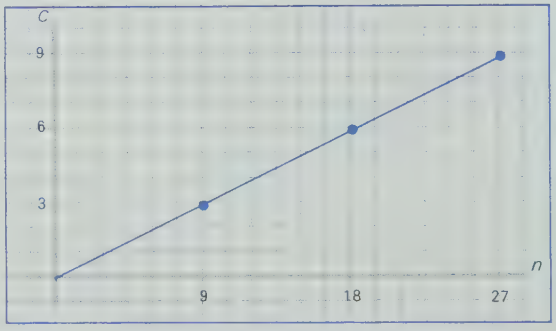
Distance	4	8	16	32	64	128
Time	1	2	4	8	16	32

3. Express each of the following as an equation.
   (*a*) $A \propto r^2$
   (*b*) $t \propto \sqrt{T}$
   (*c*) $v \propto gh^2$

4. The distance (d) that a car travels at uniform speed varies directly as the time (t) in hours taken to travel that distance. If a car travels 200 km in 2 h, find an equation representing this relationship. Sketch a graph of the equation and use the graph to determine how far the car travels in 3.5 h.

5. The length of stretch of a spring varies directly as the force applied to it. If the spring is stretched 2.5 cm by a force of 50 N, determine the equation of the relationship. Draw a graph of the equation and determine the stretch when a force of 300 N is applied.

6. Given the following graph, complete the table in your notebook and from the graph determine
   (*a*) n if $C = 3$
   (*b*) C if $n = 3$
   (*c*) n if $C = 5$



n	C
0	
3	
6	
9	
12	
15	
18	
21	
24	
27	

7. (*d*) Determine the equation of the line.

   The expense of building a highway varies directly as its length.
   (*a*) If the cost is \$2 000 000 every 5 km, how much will it cost to build a highway 57.5 km long?
   (*b*) State the equation of the relation and sketch its graph.
   (*c*) If the Minister of Highways says \$35 000 000 has been allotted to a project, how many kilometres of highway can be built?

3.(a) $A = kr^2$

4. $d = 50t$
Distance travelled in 3.5 h is 350 km

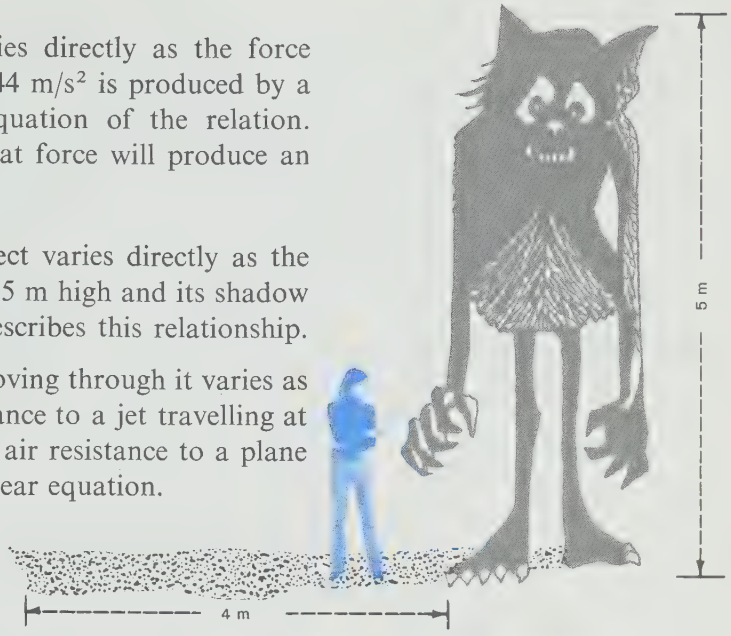
6.(a) 9
(c) 15

7.(a) \$ 23 000 000

8. The acceleration of an object varies directly as the force applied to it. If an acceleration of 44 m/s^2 is produced by a force of 154 N, determine the equation of the relation. Sketch its graph and determine what force will produce an acceleration of 60 m/s^2.

9. The length of a shadow of an object varies directly as the length of the object. If a monster is 5 m high and its shadow 4 m long, find the equation that describes this relationship.

10. The resistance of air to an object moving through it varies as the speed of the object. If the resistance to a jet travelling at 900 km/h is 5000 units, what is the air resistance to a plane travelling at 150 km/h? State the linear equation.



11. The resistance to a current in an electrical circuit varies directly as the length of the circuit. If a circuit is known to have a resistance of 10 Ω in 400 m what would be the resistance in a circuit whose length is 880 m?

Ω is the symbol for ohms

12. The circumference of a circle varies directly as its diameter. If a circle of diameter 5 cm has a circumference of 15.5 cm, calculate the following.
   (*a*) the circumference of a circle of diameter 7 cm
   (*b*) the diameter of a circle of circumference 62 cm
   (*c*) the radius of a circle whose circumference is 24.8 cm
   (*d*) the linear equation representing this relationship

13. The interest on a loan varies directly with the number of months of the loan. If the interest for one year is $54, how much interest would be owed in 3 months? State the linear relationship.

14. The volume of a sphere varies directly as the cube of its diameter. If the volume of a sphere of diameter 3 cm is 56.5 cm^3, find the volume of a cube of diameter 4 cm.

8. $a = \frac{2}{7}F$
10. $833\frac{1}{3}$ units; $r = \frac{50}{9}s$
12. (a) 21.7 cm
    (c) 4 cm
14. 133.9 cm^3

10.2 Partial Variation

In many cases the value of a variable may be made up of *two parts*. One part is a constant and the other varies directly as another variable. This type of relationship gives rise to *partial variation*.

Example 1

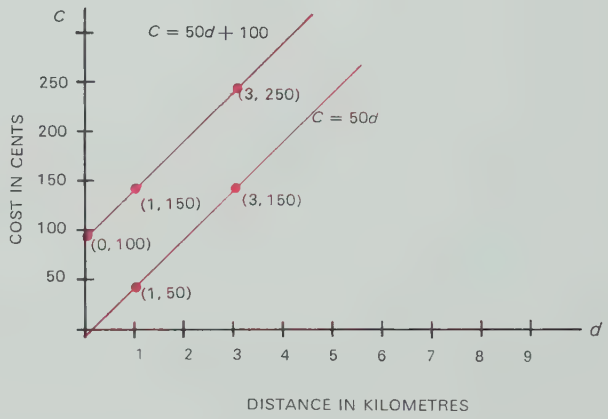
The cost of riding in a taxi depends on how far you travel plus a fixed cost. Let us say a fixed cost of \$1.00 is charged, no matter how far you go, plus a charge per kilometre of 50¢. This could be represented by the equation

$$C = 50d + 100$$

Solution

Distance (km)	0	1	3	50
Cost (cents)	$0 + 100$ $= 100$	$50(1) + 100$ $= 150$	$50(3) + 100$ $= 250$	$50(50) + 100$ $= 2600$

The graph is the same as $C = 50d$ except that 100 is added to each C value.



Example 2

A sponsor, Mr. L. Ace Fan, agrees to finance a local baseball team, the Willy-Nillies. He is told there is a fee of \$50 to cover the cost of umpires, bats and balls, and a second fee that will vary directly as the number of players on the team. The fee per player is \$8.

(*a*) Find the relationship between the cost and the number of players (n).

(*b*) Find the cost of sponsoring a team of 12 players.

(*c*) Set up a table of values and sketch the graph.

Solution

(*a*) Since the cost varies directly as the number of players plus a fixed cost,

then $C = kn + 50$.
The cost per player is \$8.
$\therefore C = 8n + 50$

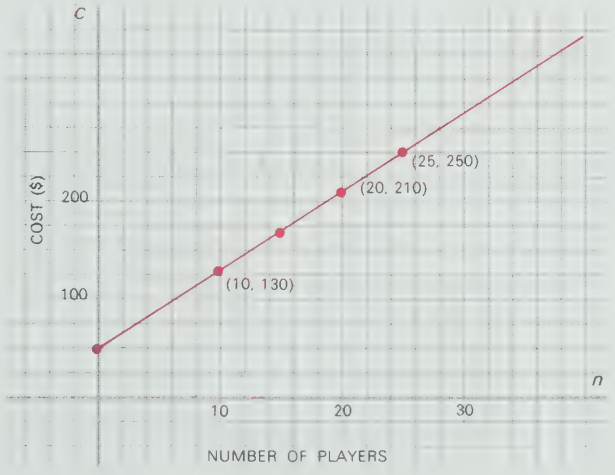
(*b*) The cost of sponsoring a team of 12:

$$\begin{aligned} C &= 8(12) + 50 \\ &= 96 + 50 \\ &= 146 \end{aligned}$$

$\therefore$ the cost is \$146.

(*c*)

Cost (\$)	50	122	130	138	146	170	210	250
No. of players	0	9	10	11	12	15	20	25



Try these.

1. The annual cost of operating a car depends on fixed costs such as insurance and licence plates, plus other costs that vary directly as the number of kilometres driven.
   (*a*) Express the relationship in the form of an equation.
   (*b*) Set up a table of values and sketch the graph for fixed costs of \$350 and a cost per kilometre of 10¢.

Compare:

$y = mx + b$
SLOPE, *y*-INTERCEPT

$C = 8n + b$
SLOPE, *y*-INTERCEPT

$C = 350 + 0.1n$
$C = \frac{1}{10}n + 350$

Example 3

The cost of printing tickets for a school music night involves a fixed charge of \$10 for the type set and a charge that varies directly as the number of tickets printed. The \$12 cost of printing 100 tickets includes the fixed cost.

(*a*) Find the equation of the relation.

(*b*) Set up a table of values and sketch a graph of the relation.

(*c*) Find the cost of printing 800 tickets.

Solution

Let C represent the total cost in dollars, f represent the fixed cost and n represent the number of tickets printed.

(*a*) The total cost equals the fixed cost plus the cost per ticket times the number of tickets.

$$\therefore C = kn + f$$

If $C = 12$, $n = 100$ and $f = 10$

$$\therefore 12 = k(100) + 10$$

$$\therefore k = \frac{2}{100}$$

$$= 0.02$$

$\therefore$ the equation of the relation is

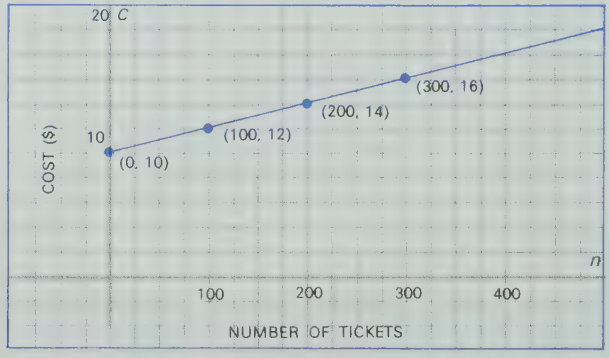
$$C = 0.02n + 10$$

$C = 0.02n + 10$

SLOPE — y-INTERCEPT

(*b*)

Cost (\$)	10	12	13	14	15	16	30
Number of tickets	0	100	150	200	250	300	1000



(*c*) When $n = 800$

$$C = (0.02)(800) + 10$$

$$= 16 + 10$$

$$= 26$$

$\therefore$ cost of printing 800 tickets is \$26.00.

Exercise 10.2

Written Solutions

1. Express the following relationships as linear equations.
   (*a*) The total value of real estate (V) is partly constant due to the location (L) and partly variable due to the number of square metres in the property (N).
   (*b*) The cost of operating a truck (C) for a day is \$45 plus \$0.10/km travelled (T).
   (*c*) The cost of printing greeting cards is made up of the sum of two items. The first is a constant and the second varies as the number of cards.
   (*d*) A number y is made up of two items. The first is a constant and the second varies as x^2.
   (*e*) The expenses of an election campaign are partly constant and partly vary as the number of election signs made.
2. The cost of printing the school yearbook will be \$500 plus \$1.75 per book.
   (*a*) Determine the equation that represents the relationship between the cost (C) and the number of books printed (n).
   (*b*) Sketch the graph.
   (*c*) How much will it cost to print 1100 books?
3. Goodtime Harry decides to throw a party for Sara Soshulset. Harry calculates the cost will be \$6.00 plus \$1.50 for each guest who arrives.
   (*a*) Express the relationship between the total cost (C) and the number of guests (n) in the form of a linear equation.
   (*b*) Sketch a graph of this relationship.
   (*c*) Determine the cost of having a party of 20 people.
4. The annual cost of operating a smowmobile depends on fixed costs of \$240 and the number of kilometres travelled. If a certain make of snowmobile costs \$0.04/km to operate, how much will it cost to drive this machine the first 500 km of the season? The second 500 km?

$V = L + kN$

$C = 0.10T + 45$

"Varies as" means varies directly as

1.(c) $C = kn + K$
(d) $y = kx^2 + K$
3.(a) $C = 1.50n + 6$
(c) \$36.00

5. The cost (C) of operating an amusement ride on the midway depends on a fixed cost (F) of \$100 per day plus the number of times (N) per day the ride must start. Determine an equation that represents this relationship if it costs \$2 each time the ride starts. If the ride starts every 3 min for 6 h on a busy day, what is the total operating cost?

6. The mass of a man varies as his height and the mass of the clothing he is wearing. A man 163 cm tall has a mass of 70 kg. Another man of about the same build is 183 cm tall. If both men are wearing 3 kg of clothing, what do you expect will be the mass of the taller man?

7. The cost of printing a school yearbook consists of two items, one of which is fixed, and a second that varies as the number of books. For 100 books, the cost is \$106.50 and for 150 books the cost is \$136.50.
   (*a*) Find the equation of the relationship.
   (*b*) Draw the graph of the relation.
   (*c*) Determine the cost of 360 books.

8. The total cost of a wedding reception is made up of fixed expenses for the rental of the premises plus an amount that varies directly as the number of guests. For 250 guests the cost is \$450, and for 150 guests the cost is \$300. Find the cost for 180 guests.

9. A number y is made up of the sum of two quantities. The first is a constant and the second varies as x. When $y = 6$, $x = 1$, and when $y = 9$, $x = 3$. Find y in terms of x.

10. Go back and solve the problem that introduced this unit, page 161.

5. $C = 100 + 2N$; \$340

7.(a) $C = 0.60n + 46.50$
(c) \$262.50

9. $y = \frac{9}{2} + \frac{3}{2}x$

10.3 Inverse Variation

Sometimes a change in one quantity brings about an opposite change in the other, that is, if one increases the other decreases. An increase in mileage, for example, decreases the value of the car. An increase in temperature decreases cooking time.

Suppose a jet airliner makes a daily flight between two cities. The following table shows the average rate on successive days and the corresponding time to complete the flight.

R(*rate in kilometres per hour*)	t(*time in hours*)
600	4
400	6
300	8
800	3

RATE HALVED, TIME DOUBLED

RATE DOUBLED, TIME HALVED

Notice that a definite relationship exists between the rate and the time throughout the table.

$$R = \frac{2400}{t} = \frac{1}{t}(2400)$$

$$\therefore \frac{R}{\frac{1}{t}} = 2400$$

$$\therefore \frac{R}{\frac{1}{t}} = \text{a constant or } R \text{ varies directly as } \frac{1}{t}.$$

In this case rate R is said to *vary inversely* as the time (t) or $R \propto \frac{1}{t}$.

If two variables are related so that one divided by the reciprocal of the other produces a constant, that is, their product is constant, then one *varies inversely* as the other.

$$\text{If } R \propto \frac{1}{t},$$

$$\text{then } R = k \cdot \frac{1}{t}$$

$$\therefore R = \frac{k}{t}$$

Recall this ?

$\frac{7}{\frac{1}{2}} = 7 \times \frac{2}{1}$

$= 14$

If R increases then t decreases.

Example 1

The current (I) in an electrical circuit varies inversely as the resistance (R) in the circuit. Find the current when the resistance is 100 000 Ω if we know the resistance is 50 Ω when the current is 2 A.

Solution

$$I \propto \frac{1}{R}$$

$$\therefore I = \frac{k}{R}$$

$$\text{or} \quad IR = k$$

$$\text{When } I = 2, R = 50.$$

$$\therefore k = 2 \times 50$$

$$= 100$$

$$\therefore \text{when } R = 100\,000$$

$$\therefore I = \frac{100}{100\,000} = \frac{1}{1000} = 0.001$$

$\therefore$ the current is 0.001 A when the resistance is 100 000 Ω.

Example 2

If a string is under constant tension, its rate of vibration varies inversely as its length. A string 60 cm long has a frequency of 250 Hz. What length of string will have a frequency of 600 Hz if the tension is the same?

Solution

$$V \propto \frac{1}{L}$$

$$\therefore V = \frac{k}{L}$$

$$\text{When } V = 250, L = 60$$

$$\therefore 250 = \frac{k}{60}$$

$$\therefore k = 250 \times 60$$

$$= 15\,000$$

kHz is the symbol for kilohertz

When $V = 600$

$$\therefore 600 = \frac{15\,000}{L}$$

$$L = 25$$

$\therefore$ the string is 25 cm long.

Exercise 10.3

Discussion

1. Express each of the following as an equation.

   (*a*) $P \propto \frac{1}{V}$ (*c*) $f \propto \frac{1}{L}$

   (*b*) $A \propto \frac{1}{d}$ (*d*) $I \propto \frac{1}{R}$

2. The value of an automobile varies inversely as its age. State a variation equation to show this.

3. State a variation equation that shows that the attraction between two objects varies inversely as the square of their distance apart.

Written Solutions

4. The force of attraction (F) between two spherical objects varies inversely as the square of the distance between their centres. State the equation of variation and find the force of attraction when the objects are 15 cm apart if the force is 10 when the objects are 5 cm apart.

5. According to Boyle's Law the volume (V) of a gas varies inversely as its pressure (k) when the temperature is constant. State the equation of the relationship and find the volume of gas at 72 kPa if the volume at 76 kPa is 30 ℓ

6. The amount of heat (A) received from a radiator varies inversely as the square of the distance (d) from it. If a person 8 m from the radiator receives 10 units of heat, how much heat is received by a person 2 m from the radiator?

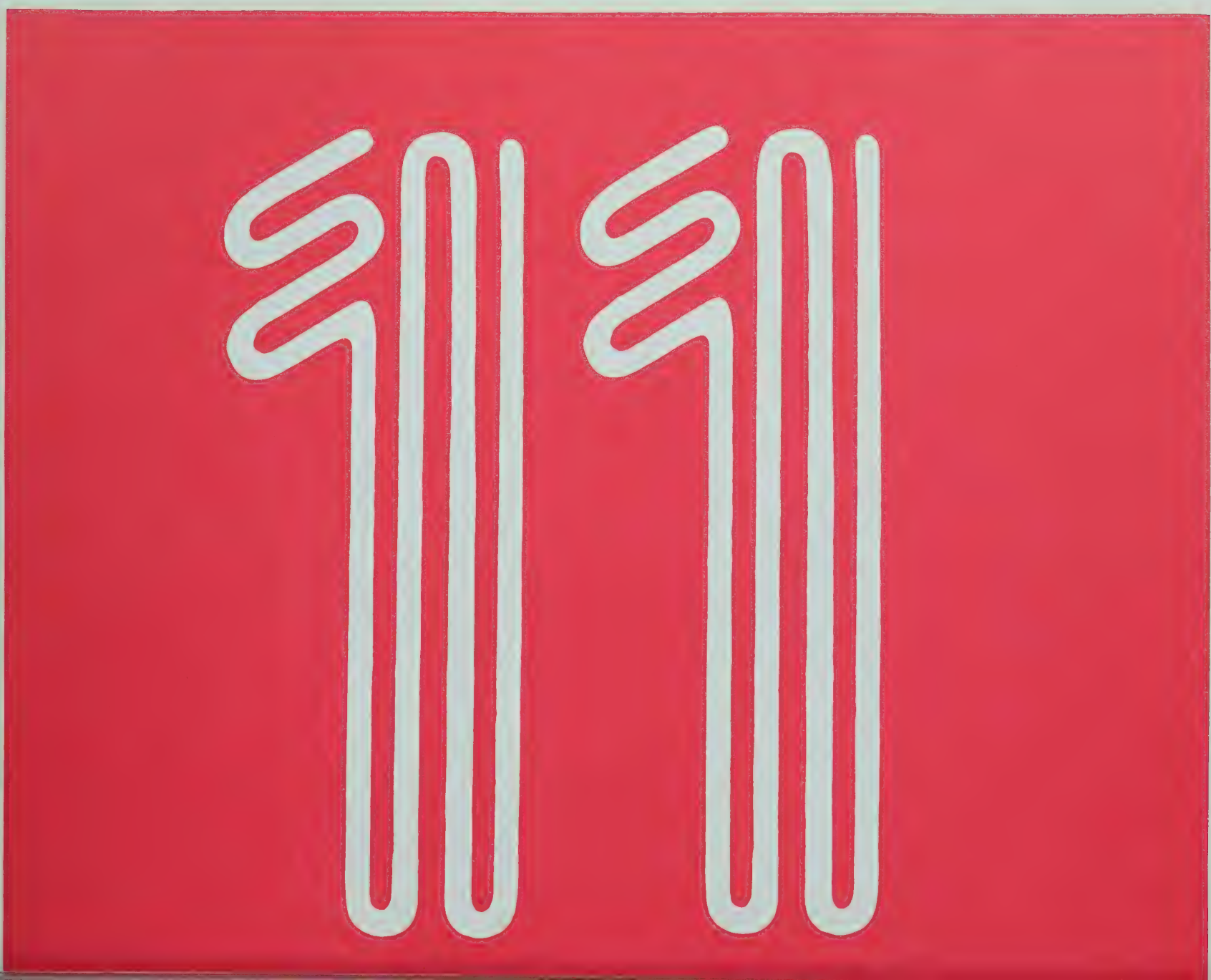
7. When the teeth of one gear mesh with those of another, the angular velocity of each gear varies inversely as the number of teeth. If one gear has 36 teeth and is rotating at 48 r/min, how many teeth has the other if it is rotating at 216 r/min?

Pa is the symbol for pascals.

4. $F = \frac{k}{d^2}$; $\frac{10}{9}$ units

6. 160 units

7. 8 teeth

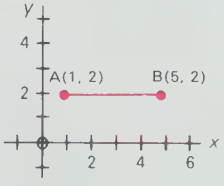


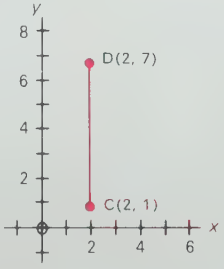
Unit 11
Coordinate Geometry

11.1 Lengths of Line Segments Parallel to the Axes

A Eventually we would like to be able to determine the length of any line segment by using the coordinates of the end points. Before we do this let's first find the lengths of line segments parallel to the axes.

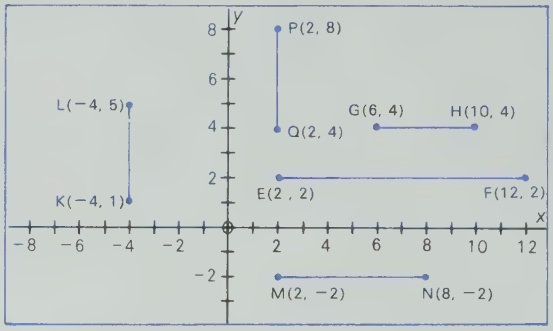
1. Compare the direction of line segment AB and the x axis in the diagram. Use the coordinates of A and B to justify your answer.
2. What is the length of AB?
3. Compare the direction of line segment CD and the y axis in the diagram.
4. Use the coordinates of C and D to justify your answer. What is the length of CD?





∥ means "parallel to".

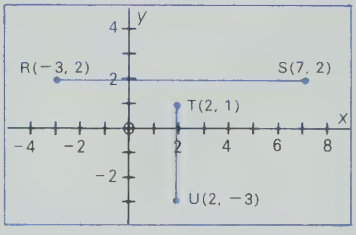
5. Calculate the lengths of each of the line segments in the following diagram.



B Sometimes the line segments extend to another quadrant.

Example

What are the lengths of line segments RS and TU in the following diagram?

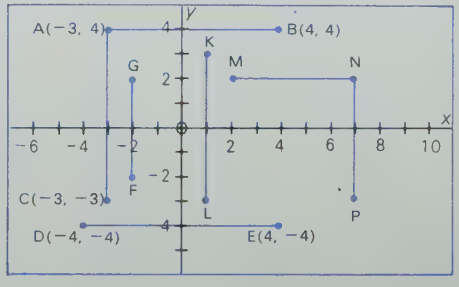


Solution

$$\begin{aligned}\text{Length of RS} &= 7 - (-3)\\ &= 7 + 3\\ &= 10 \text{ units}\end{aligned}$$

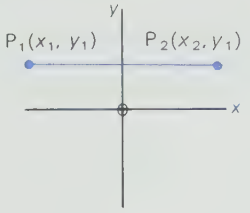
$$\begin{aligned}\text{Length of TU} &= 1 - (-3)\\ &= 1 + 3\\ &= 4 \text{ units}\end{aligned}$$

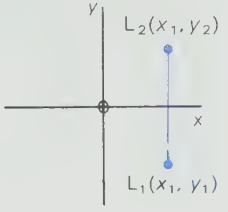
1. Calculate the lengths of the line segments in the following diagram.



C Let's develop a relationship in general terms.

1. How do you know the line P_1P_2 is parallel to the x axis?
2. Express the length of P_1P_2 using the coordinates of P_1 and P_2.
3. Why can you say that line L_1L_2 is parallel to the y axis?
4. Express the length of L_1L_2 using the coordinates of L_1 and L_2.





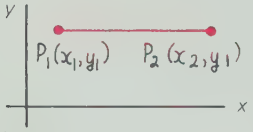
5. Copy and complete the following table in your notebook.

Line Segment	*Coordinates*	*Line Segment Parallel to x axis or y axis*	*Length*
M_1M_2	$(x_1, y_1), (x_2, y_1)$	x axis	$x_2 - x_1$
K_1K_2	$(x_1, y_1), (x_1, y_2)$		
L_1L_2	$(x_1, y_1), (\ \ , y_1)$	x axis	$x_2 - x_1$
P_1P_2	$(x_1, \ \ ), (x_1, \ \ )$		$y_2 - y_1$
S_1S_2	$(x_1, y_1), (x_1, y_2)$	y axis	
T_1T_2	$(\ \ , y_1), (\ \ , y_1)$		$x_2 - x_1$

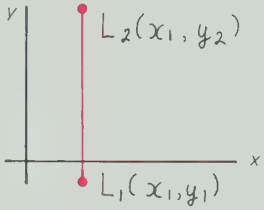
Exercise 11.1

Discussion

1. Select pairs of points that make up line segments parallel to the x axis.
   (*a*) $(3, 2), (-3, 2)$
   (*b*) $(-4, 3), (-4, 8)$
   (*c*) $(5, -5), (-5, -5)$
   (*d*) $(8, -2), (-2, 8)$
   (*e*) $(3, 6), (8, 6)$
   (*f*) $(-7, 4), (4, 4)$



Length of $P_1P_2 = x_2 - x_1$



Length of $L_1L_2 = y_2 - y_1$

Make a rough sketch if you need to.

3.(a) 8 (f) 6
(c) 9 (h) 10
(e) 7 (j) 6

2. Select pairs of points that make up line segments parallel to the y axis.

(*a*) (−3, 4), (3, 4) (*d*) (4, 8), (8, 4)
(*b*) (2, 5), (2, −5) (*e*) (−3, 7), (−3, −7)
(*c*) (3, 7), (3, 11) (*f*) (6, 2), (−6, −2)

Written Solutions

3. Determine the lengths of the line segments joining the following pairs of points.

(*a*) A(2, 0), B(10, 0) (*f*) X(4, −7), Y(4, −1)
(*b*) C(−3, 1), D(5, 1) (*g*) T(−2, 8), V(−6, 8)
(*c*) E(−2, −4), F(−2, 5) (*h*) K(0, 5), L(0, −5)
(*d*) P(−3, −3), Q(−3, 9) (*i*) G(1, −10), H(1, −1)
(*e*) R(−3, −2), S(−10, −2) (*j*) M(−5, 6), N(−11, 6)

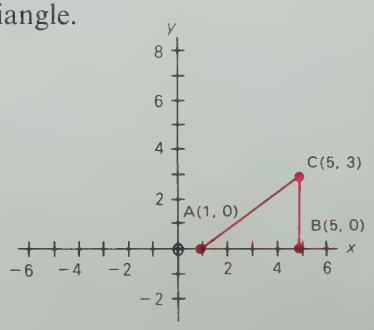
4. In each of the following cases, point A is midway between point B and point C. Copy and complete the table.

Coordinates of Point A	*Coordinates of*		*Parallel Axis*	*Length of AB and AC*
	Point B	*Point C*		
(2, 4)			*y* axis	8
(−2, 5)			*x* axis	7
(3, −1)			*y* axis	11
(4, 7)			*x* axis	3
(1, −1)			*x* axis	5
(5, −8)			*y* axis	6
(0, 4)			*y* axis	9

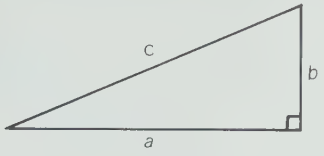
11.2 Lengths of Line Segments Joining Any Two Points

A It is now possible to study lengths of line segments by considering the lengths of the sides of a special triangle.

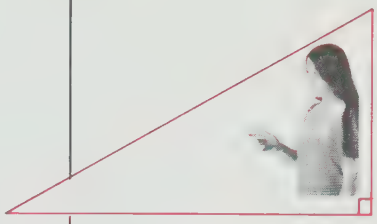
1. In the diagram at the right, calculate the lengths of AB and BC.
2. What is the size of angle CBA?
3. What kind of triangle is △ABC?



B In order to find the length of the hypotenuse of a right triangle, it is necessary to recall the Pythagorean Theorem.

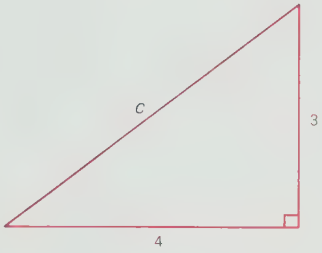


$$c^2 = a^2 + b^2$$



Example

Given the following diagram of a right triangle, determine the length of c.



Solution

Since $c^2 = 4^2 + 3^2$

$\therefore c^2 = 16 + 9$

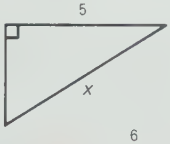
$= 25$

$\therefore c = 5$

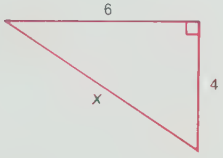
Now try these.

1. Determine the length of x in each of the following triangles.

(a)

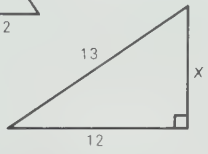


(b)

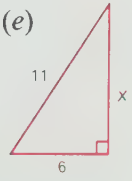


(c)

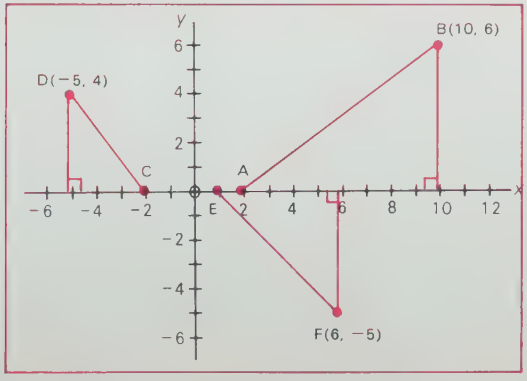
(d)



(e)

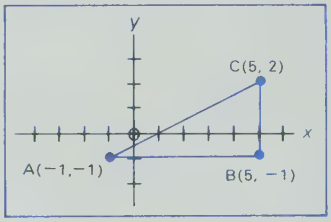


2. Determine the length of the named line segments in the diagram.



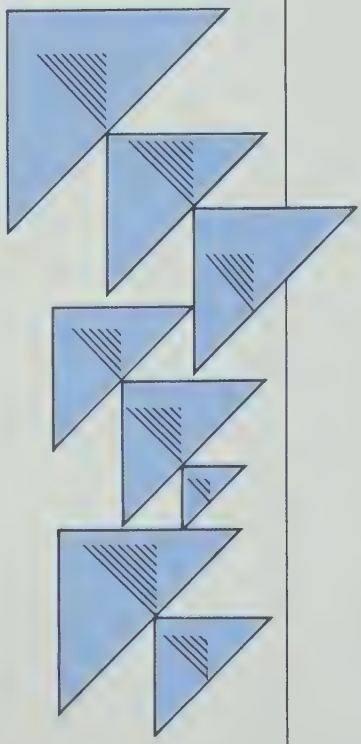
See the tables for $\sqrt{34} \doteq 5 \cdot 83$

C Let us now consider the line segment AC.

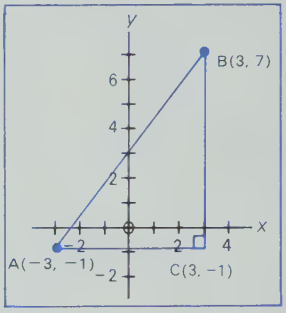


AB $\parallel$ x axis
BC $\parallel$ y axis
Why?
$\therefore$ $\triangle$ABC is a right triangle.

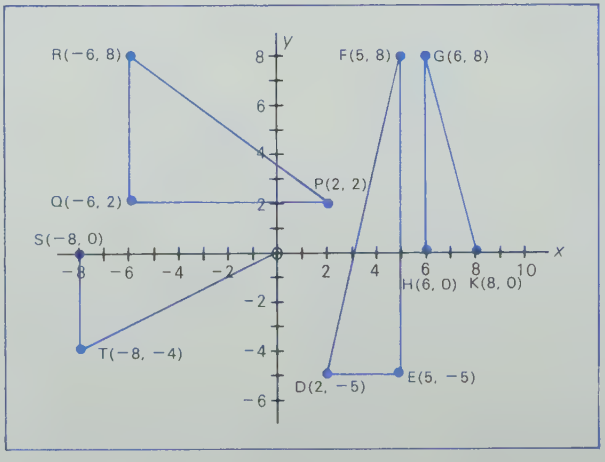
Length of AB $= 5 - (-1) = 6$
Length of BC $= 2 - (-1) = 3$
$\therefore$ Length of AC $= \sqrt{6^2 + 3^2}$
$= \sqrt{36 + 9}$
$= \sqrt{45}$
$= \sqrt{9 \times 5}$
$= 3\sqrt{5}$



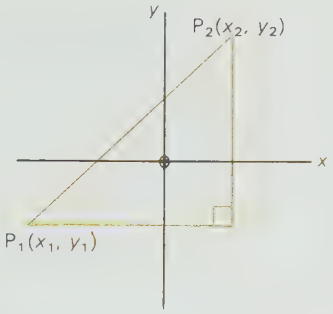
1. In the diagram that follows, AC is parallel to the x axis and BC is parallel to the y axis.
   (*a*) Determine the coordinates of C.
   (*b*) Calculate the lengths of AC, BC and AB.



2. Determine the length of each side of the triangles in the following diagram.



D As an aid in the calculation of lengths in coordinate geometry, it is useful to develop a formula from a general case. With this in mind, calculate the length of line segment P_1P_2.



$P_1P_2 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Exercise 11.2

Written Solutions

1. State an expression for the distance between each of the following pairs of points.
   (*a*) (0, 0), (−3, −4)
   (*b*) (5, 12), (0, 0)
   (*c*) (−10, −6), (−2, 0)
   (*d*) (8, 1), (−2, −6)
   (*e*) (1, −4), (9, 14)
   (*f*) (−2, 5), (3, −1)
   (*g*) (1, 6), (8, −2)
   (*h*) (−3, 4), (9, 0)

2. Calculate the lengths of line segments whose end points are as follows.
   (*a*) (−6, 3) and (4, 5)
   (*b*) (−2, −5) and (7, −4)
   (*c*) (3, 3) and (−1, −1)
   (*d*) (1, −4) and (1, −6)
   (*e*) (0, 0) and (−8, −6)
   (*f*) (−4, 7) and (3, −2)

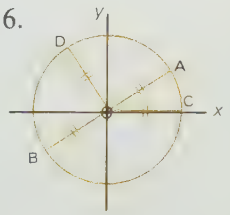
3. Find the perimeter of the figures.
   (*a*) A(−4, 2), B(7, 2) and C(−4, 6)
   (*b*) P(8, 0), Q(4, 3) and R(0, 0)

4. Find the value of x so that the length of the segment joining the origin to P(x, 4) is 5.

5. Find the value of y so that the length of the segment joining the origin to R(5, y) is 13.

6. Prove that the points A(8, 6), B(−6, 8), C(10, 0), and D(−8, −6) lie on the circle with centre O(0, 0).

7. Prove by distances that, if the following points are joined, they form an isosceles triangle.
   (*a*) A(2, 2), B(5, 10), C(8, 2)
   (*b*) P(−4, 4), Q(−20, 10), R(−4, 16)
   (*c*) D(−3, −1), E(−4, −5), F(−2, −5)
   (*d*) K(1, −2), L(1, −6), M(6, −4)

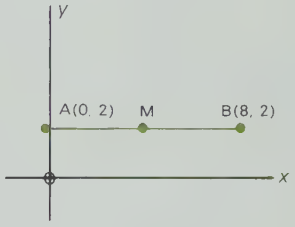
6.


1.(a) $\sqrt{(-3)^2+(-4)^2}$
(c) $\sqrt{8^2+6^2}$
(e) $\sqrt{8^2+18^2}$
(g) $\sqrt{7^2+(-8)^2}$
2.(b) $\sqrt{82}$ (e) 10
3.(b) 18
5. ±12

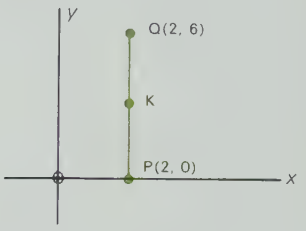
11.3 Midpoints of a Line Segment

A Once we can determine the length of a line segment, it is often necessary and useful to find the coordinates of the midpoint of the line.

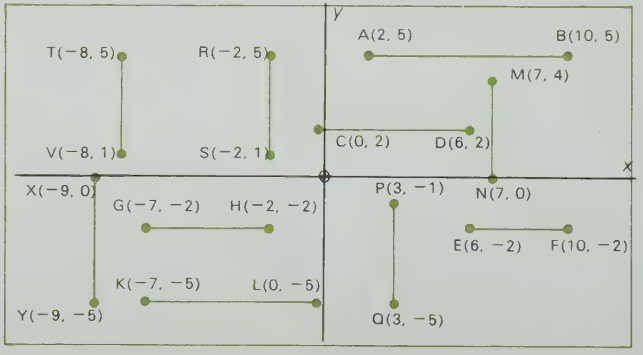
1. First consider a line segment AB. If M is the midpoint, then it is obvious that
$$AM = MB = \tfrac{1}{2} AB.$$
2. In the diagram, M is the midpoint of AB.
   (*a*) What is the y coordinate of M?
   (*b*) What is the x coordinate of M?
   (*c*) Verify your conclusion by showing that $AM = MB = \tfrac{1}{2} AB$.



3. In the following diagram, K is the midpoint of PQ.
   (*a*) What is the x coordinate of K?
   (*b*) What is the y coordinate of K?
   (*c*) Verify your conclusion by proving that $PK = KQ = \tfrac{1}{2} PQ$.



4. Determine the coordinates of the midpoints of each of the following line segments.



B So far we have looked only at special line segments. The end points of each line were in the same quadrant. Now let's study and determine the midpoints of line segments whose end points do not both lie in the same quadrant.

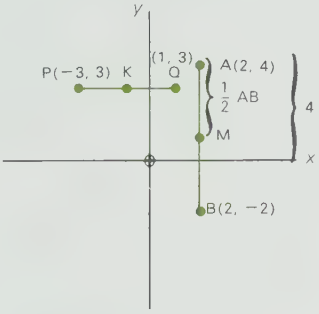
Example

In the diagram at the right, M is the midpoint of AB.

(*a*) What is the x coordinate of M?

(*b*) What is the y coordinate?

(*c*) Show that $AM = MB = \frac{1}{2}AB$.



Solution

(*a*) x coordinate of M is 2.

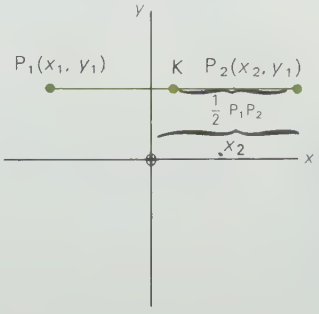
(*b*) y coordinate $= 4 - \frac{1}{2}AB$
$= 4 - \frac{1}{2}(4 + 2)$
$= 4 - 3$
$= 1$

(*c*) $AM = 4 - 1$
$= 3$
$MB = 1 + 2$
$= 3$
$\frac{1}{2}AB = \frac{1}{2}(4 + 2)$
$= 3$

1. K is the midpoint of PQ.
   (*a*) What is the y coordinate of K?
   (*b*) What is the x coordinate of K?

C Let's try to find an expression for determining the coordinates of the midpoint of a line parallel to the x axis.

Let K represent the midpoint.



y coordinate of K $= y_1$

x coordinate of K $= x_2 - \frac{1}{2}P_1P_2$
$= x_2 - \frac{1}{2}(x_2 - x_1)$
$= x_2 - \frac{1}{2}x_2 + \frac{1}{2}x_1$
$= \frac{1}{2}x_2 + \frac{1}{2}x_1$
$= \frac{x_2 + x_1}{2}$

$\therefore$ coordinates of K are $\left(\frac{x_1 + x_2}{2}, y_1\right)$.

|| to the x axis means same y values.

$P_1P_2 = x_2 - x_1$

II to y axis means same x values.

$P_1P_2 = y_2 - y_1$

We can also find an expression for the coordinates of the midpoint of a line parallel to the y axis.

Let M represent the midpoint.

x coordinate of M $= x_1$.

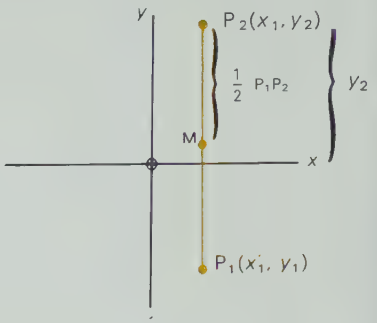
y coordinate of M $= y_2 - \frac{1}{2} P_1P_2$

$= y_2 - \frac{1}{2}(y_2 - y_1)$

$= y_2 - \frac{1}{2} y_2 + \frac{1}{2} y_1$

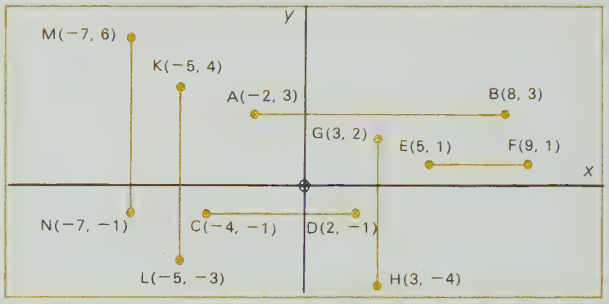
$= \frac{1}{2} y_2 + \frac{1}{2} y_1$

$= \frac{y_2 + y_1}{2}$



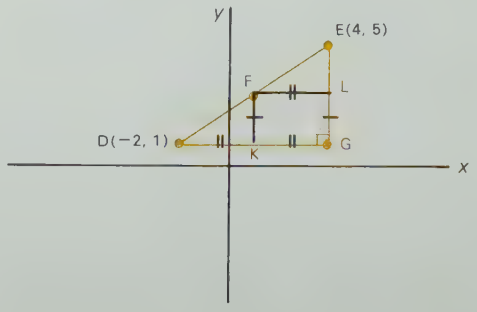
$\therefore$ coordinates of M are $\left(x_1, \frac{y_1 + y_2}{2}\right)$.

1. In the following diagram, determine the coordinates of the midpoints of each of the line segments.



D So far we have worked only with line segments that are parallel to the axes. We want to be able to find the midpoint of any line segment. In the following diagram, consider also DE whose midpoint is F.

1. Draw EG $||$ y axis and DG $||$ x axis.
2. G has coordinates (4, 1). Why?



3. Find the coordinates of K, the midpoint of DG.

$$x \text{ coordinate} = \frac{4 + (-2)}{2}$$

$$= \frac{2}{2}$$

$$= 1$$

$\therefore$ coordinates of K are (1, 1).

Midpoint of a line segment $\parallel x$ axis: $\left(\frac{x_1 + x_2}{2}, y_1\right)$

4. Find the coordinates of L, the midpoint of EG.

$$x \text{ coordinate} = 4$$

$$y \text{ coordinate} = \frac{5 + 1}{2}$$

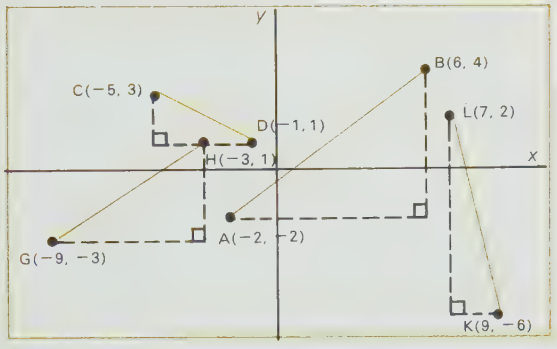
$$= 3$$

$\therefore$ coordinates of L are (4, 3).

Midpoint of a line segment $\parallel y$ axis: $\left(x_1, \frac{y_1 + y_2}{2}\right)$

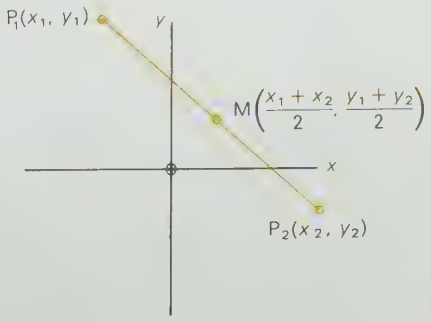
If we think of DKF and FLE as "steps," we can see that the x coordinate of F is the same as the x coordinate of K. Also the y coordinate of F is the same as the y coordinate of L. The coordinates of F are (1, 3).

5. Determine the midpoints of each of the following line segments.



Summary

The coordinates of the midpoint M of the line segment joining $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

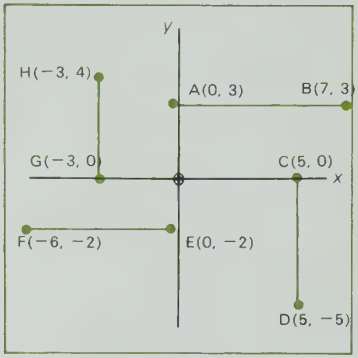


Exercise 11.3

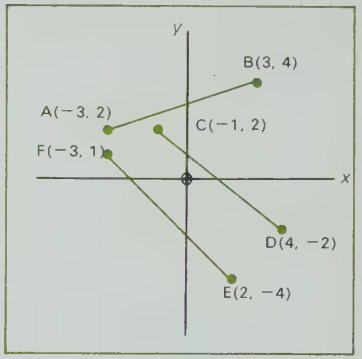
Written Solutions

1. For each line segment in the following diagrams determine the coordinates of the midpoint.

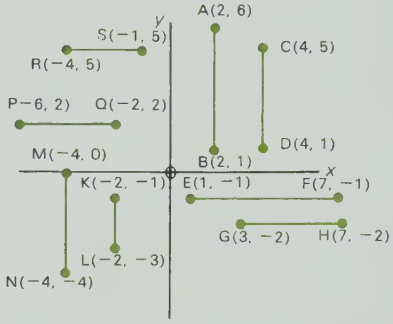
(a)



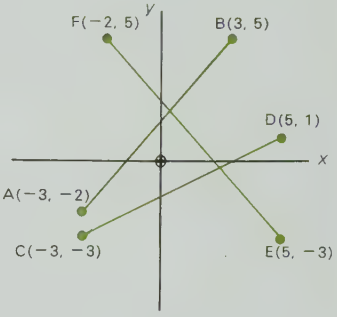
(d)



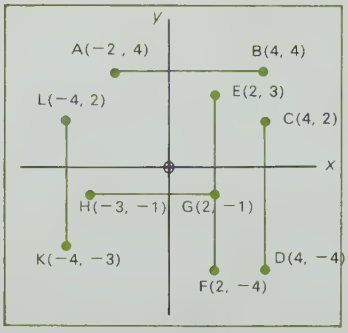
(b)



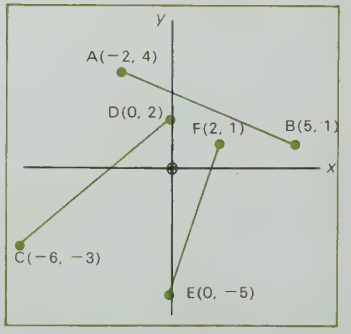
(e)



(c)



(f)



1.(a) AB $(3\frac{1}{2}, 3)$
GH $(-3, 2)$
(b) CD $(4, 3)$
GH $(5, -2)$
PQ $(-4, 2)$
(c) CD $(4, -1)$
KL $(-4, -\frac{1}{2})$
(d) AB $(0, 3)$
(e) AB $(0, \frac{3}{2})$
(f) EF $(1, -2)$

2.(b) $(1, 5)$ (d) $(-4, -3)$
(c) $(-4, -2)$

2. Find the coordinates of the midpoints of line segments with the following end points.
   (*a*) (2, −5) and (3, −4)
   (*b*) (−6, 5) and (8, 5)
   (*c*) (−4, −7) and (−4, 3)
   (*d*) (0, −3) and (−8, −3)
   (*e*) (a, b) and (c, d)
   (*f*) $\left(\frac{a}{4}, \frac{b}{5}\right)$ and $\left(\frac{a}{2}, \frac{b}{2}\right)$

3. (*a*) Find the midpoints of the sides of triangles with the given vertices.
   (i) A(−2, −3), B(6, −3), C(4, 5)
   (ii) J(4, 2), K(9, 14), L(−1, 14)
   (*b*) Find the lengths of the *medians* of the triangles in part (*a*).

4. (*a*) Show that the y axis bisects the line segment joining A(2, −7) and B(−2, 1).
   (*b*) Show that the x axis bisects the line segment joining A(−5, 8) and B(8, −8).

5. Find the coordinates of the centre of a circle whose diameter has end points (−3, −2) and (3, 2).

6. Show that the diagonals of the quadrilateral with end points A(−5, 2), B(3, 4), C(3, 1) and D(−5, −1) bisect each other.

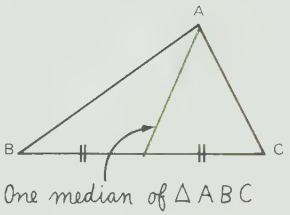
7. The points A(−1, −1), B(−3, −5) and C(−7, −3) are the vertices of a triangle.
   (*a*) Find the coordinates of the midpoints of AC and AB.
   (*b*) Prove that the line joining these midpoints equals $\frac{1}{2}$ BC.
   (*c*) Prove that the line joining the midpoints is parallel to BC.

8. If (5, 6) is one end point of a line segment and the coordinates of the midpoint are (1, 5), find the other end point.

9. If the coordinates of one end point of a line segment are (−3, −7) and the coordinates of the midpoint are (0, 0), what are the coordinates of the other end point?

10. If the end points of a diameter of a circle are (−5, 3) and (−1, −3), find the coordinates of the centre of the circle.

11. The centre of a circle has coordinates (3, −2). One end of a diameter has coordinates (5, 3). Find the coordinates of the other end point of that diameter.



One median of $\triangle ABC$

Parallel lines have the same slope.

4.(a) Bisects at (0, −3)
(b) Bisects at $(\frac{3}{2}, 0)$
5. (0, 0)
8. (−3, 4)
9. (3, 7)

Unit 12
Second Degree Relations

Possibly because of its symmetry or because of the shape of the sun and the moon, the circle was considered by early mathematicians to be the perfect curve. Circular motion forms the basis of many mechanical devices and so exploits the geometrical properties of the circle. We will look at some of these properties using coordinate geometry.

We will also examine the graphs of quadratic relations that form the curve of a special shape called the *parabola*. Parabolas also have many practical applications in today's world. For example, the path of a projectile and the outline of a headlight reflector are parabolas. Again the study will be made using coordinate geometry.

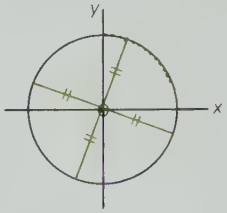
12.1 The Circle as the Graph of a Relation

A Recall these characteristics of the figure called a *circle*.

A circle is a set of points.

All points are the same distance from the centre.

The *radius* is the line segment joining the centre to any point on the circle. The word radius is also used to describe the length of the segment.



To describe or to draw a circle we need to know only its centre and the length of its radius.

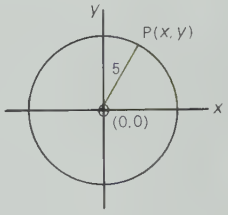
B We can find the equation of a circle by making use of the preceding ideas.

Example

State an equation of the circle with centre at the given point (0, 0) and radius of 5 units.

Solution

Since P is only one point of a set of points that make up the circle, let it have the general coordinates (x, y).



Length of OP $= 5$

Also OP $= \sqrt{(x-0)^2 + (y-0)^2}$

$= \sqrt{x^2 + y^2}$

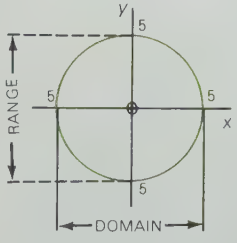
$\therefore \sqrt{x^2 + y^2} = 5$

$\therefore x^2 + y^2 = 25$

$\therefore$ an equation of the circle with centre (0, 0) and radius 5 units is $x^2 + y^2 = 25$.

Remember?

$P_1P_2 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$



Recall that the *domain* of a relation is the *horizontal extent* of its graph, and the *range* of a relation is the *vertical extent* of the graph.

Notice that for any point (x, y) on our circle the value of x extends from -5 to 5.

$\therefore$ the domain is $\{x \mid -5 \leqslant x \leqslant 5\}$.

Also the value of y extends from -5 to 5.

$\therefore$ the range is $\{y \mid -5 \leqslant y \leqslant 5\}$.

1. Write an equation for each of the following circles.
   (*a*) centre (0, 0) and radius 2 units
   (*b*) centre (0, 0) and radius 10 units
   (*c*) centre (0, 0) and radius $\sqrt{3}$ units

2. State the domain and range for each part of Question 1.

3. Find an equation of the circle with centre (0, 0) and radius r.

$$x^2 + y^2 = (\text{radius})^2$$

C Given the equation of a circle, we should be able to determine its radius. For example, if $x^2 + y^2 = 9$, we know this can be written $x^2 + y^2 = (3)^2$ which is an equation for a circle with its centre at (0, 0) and with a radius of 3 units.

1. Determine the centre and radius of each of the following circles.
   (*a*) $x^2 + y^2 = 36$
   (*b*) $x^2 + y^2 = \frac{1}{9}$
   (*c*) $x^2 + y^2 = 3$

2. State the domain and range for each part of Question 1.

Exercise 12.1

Written Solutions

1. Write an equation for each of the following circles and determine the domain and range.
   (*a*) centre (0, 0), radius 9
   (*b*) centre (0, 0), radius $\sqrt{5}$
   (*c*) centre (0, 0), radius $\frac{2}{3}$
   (*d*) centre (0, 0), radius $2\sqrt{5}$
   (*e*) centre (0, 0), radius 1
   (*f*) centre (0, 0), radius $\frac{1}{9}$

2. Determine the radius, domain, and range of each of the following circles,
   (*a*) $x^2 + y^2 = 36$
   (*b*) $x^2 + y^2 = 5$
   (*c*) $x^2 + y^2 = 1$
   (*d*) $x^2 + y^2 = \frac{4}{9}$
   (*e*) $x^2 + y^2 = \frac{1}{4}$
   (*f*) $4x^2 + 4y^2 = 9$
   (*g*) $5x^2 + 5y^2 = 20$
   (*h*) $3x^2 + 3y^2 = 48$
   (*i*) $7x^2 + 7y^2 = 12$
   (*j*) $x^2 + y^2 = \frac{3}{16}$

3. Rewrite each equation in Question 2 in the form $\sqrt{x^2 + y^2} = (\ \ )$, and describe each circle as a set of points a given distance from a given point.

$x^2 + y^2 = r^2$

1.(a) $x^2 + y^2 = 81$; $\{x | -9 \leq x \leq 9\}$; $\{y | -9 \leq y \leq 9\}$
(e) $x^2 + y^2 = 1$; $\{x | -1 \leq x \leq 1\}$; $\{y | -1 \leq y \leq 1\}$
2.(a) 6; $\{x | -6 \leq x \leq 6\}$; $\{y | -6 \leq y \leq 6\}$
(d) $\frac{2}{3}$; $\{x | -\frac{2}{3} \leq x \leq \frac{2}{3}\}$; $\{y | -\frac{2}{3} \leq y \leq \frac{2}{3}\}$
(g) 2; $\{x | -2 \leq x \leq 2\}$; $\{y | -2 \leq y \leq 2\}$
(h) 4; $\{x | -4 \leq x \leq 4\}$; $\{y | -4 \leq y \leq 4\}$

Find the radius first!

4.(a) $x^2 + y^2 = 25$
(d) $x^2 + y^2 = 4$
(e) $x^2 + y^2 = 100$
(h) $x^2 + y^2 = 64$

5.(b) $x^2 + y^2 = 29$
(e) $x^2 + y^2 = 34$

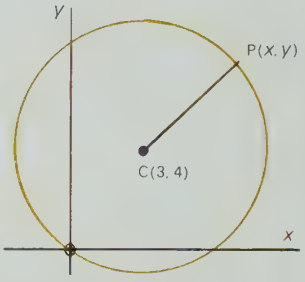
4. Write equations of the following circles that have the origin as centre and pass through the given points.
   (*a*) (5, 0)
   (*b*) $(0, -\sqrt{7})$
   (*c*) (2, 1)
   (*d*) $(\sqrt{3}, 1)$
   (*e*) (−8, 6)
   (*f*) (3, 4)
   (*g*) $(2\sqrt{2}, -1)$
   (*h*) $(6, 2\sqrt{7})$

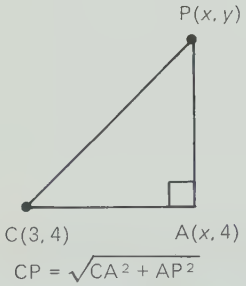
5. Determine equations of the following circles with centre the origin and end points of a diameter as given.
   (*a*) (3, 4), (−3, −4)
   (*b*) (5, 2), (−5, −2)
   (*c*) (−3, 1), (3, −1)
   (*d*) (−5, −6), (5, 6)
   (*e*) (5, −3), (−5, 3)
   (*f*) (7, −5), (−7, 5)

12.2 Equation of Circles, Centre Not at the Origin

Let's find out how to develop the equation of a circle whose centre is *not* at the origin.

Consider a circle whose centre is at (3, 4) and with a radius of 5 units.





If P(x, y) is any point on the circle,

$\text{CP} = \sqrt{(x-3)^2 + (y-4)^2}$

Also $\text{CP} = 5$

$\therefore \sqrt{(x-3)^2 + (y-4)^2} = 5$

$\therefore$ the equation of the circle, centre (3, 4) and radius 5, is $(x-3)^2 + (y-4)^2 = 25$.

DOMAIN:
all x values.

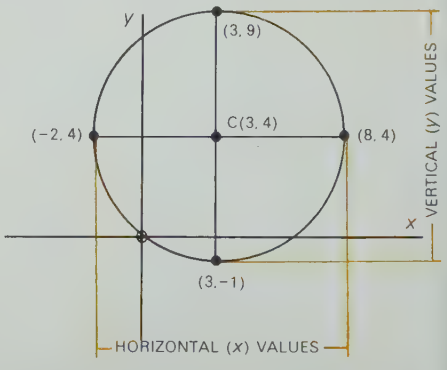
RANGE:
all y values.

By looking for the horizontal extent of the graph, we can find the domain.

$\therefore$ domain is $\{x \mid -2 \leqslant x \leqslant 8\}$

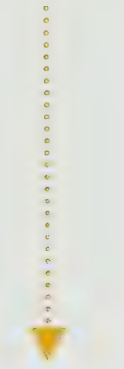
By looking for the vertical extent of the graph, we can find the range.

$\therefore$ range is $\{y \mid -1 \leqslant y \leqslant 9\}$



An equation of a circle with centre at (h, k) and radius r units is $(x-h)^2 + (y-k)^2 = r^2$.

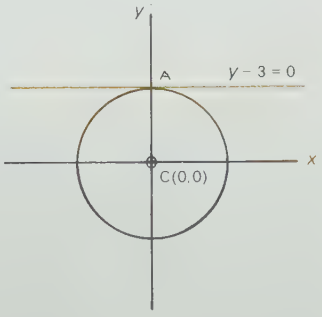
1. Determine an equation of a circle with a radius of 5 units and centre at (2, 3).
2. State the domain and range of the circle in Question 1.
3. Given the equation of a circle $(x + 2)^2 + (y + 3)^2 = 16$, state the coordinates of the centre and its radius.
4. State the domain and range of the circle in Question 3.



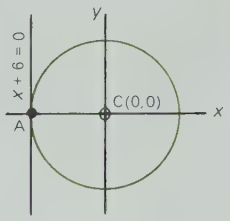
Watch out! Centre is not at the origin!

Exercise 12.2

1. State an equation for each circle whose centre and radius are given.
   (*a*) C (3, 2); 5 units
   (*b*) C (2, 0); 3 units
   (*c*) C (1, 2); $\sqrt{2}$ units
   (*d*) C (−1, 1); 2 units
   (*e*) C (−5, 2); 7 units
   (*f*) C (−3, 4); 5 units
   (*g*) C $(\frac{1}{2}, -\frac{1}{2})$; 11 units
2. State the centre, radius, domain, and range of each of the following circles.
   (*a*) $(x - 5)^2 + (y - 4)^2 = 16$
   (*b*) $(x - 0)^2 + (y - 0)^2 = 25$
   (*c*) $(x + 5)^2 + (y + 4)^2 = 16$
   (*d*) $(x - 3)^2 + (y + 2)^2 = 9$
   (*e*) $(x + 5)^2 + (y - 7)^2 = \frac{1}{4}$
   (*f*) $x^2 + (y - 2)^2 = 49$
   (*g*) $(x - 2)^2 + y^2 = 1$
   (*h*) $(x + 3)^2 + (y + 4)^2 = 36$
3. Give the radius and equation of each of the following circles.
   (*a*) centre (3, 4) and passing through the origin
   (*b*) centre (4, 9) and passing through (−2, 1)
4. The given circle has centre (0, 0) and touches the line $y - 3 = 0$.
   (*a*) What are the coordinates of A?
   (*b*) What angle does CA make with the line $y - 3 = 0$?
   (*c*) What is the radius of the circle?



5. The given circle has centre $(0, 0)$ and touches the line $x + 6 = 0$.
   (*a*) What are the coordinates of A?
   (*b*) What angle does CA make with the line $x + 6 = 0$?
   (*c*) What is the radius of the circle?



A *tangent* is a line perpendicular to a radius at the circumference of the circle. The tangent meets the circle at the *point of contact*.

6. Given that the centre of a circle is $(3, 4)$ and a tangent to the circle has the equation $y = 8$.
   (*a*) Find the point of contact.
   (*b*) Find the radius of the circle.
   (*c*) State the equation of the circle.

7. A tangent to a circle has the equation $x = 0$ and the centre of the circle is $(-3, 2)$.
   (*a*) Find the point of contact.
   (*b*) Find the radius of the circle.
   (*c*) State the equation of the circle.

8. Find the equation of a circle with centre $(2, -4)$ that has as a tangent the line $y = 1$.

9. The tangent to a circle with centre $(-5, -7)$ is defined by $x = -2$. Find the equation of the circle.

10. The line segment joining $(5, -7)$ and $(1, -3)$ is a diameter of a circle.
    (*a*) Find the centre.
    (*b*) Find the radius.
    (*c*) Find the equation.

11. Repeat Question 10 using the given points.
    (*a*) $(6, 9)$ and $(-2, -3)$
    (*b*) $(-2, 3)$ and $(-4, -1)$

12. A circle passes through the origin O, has radius 5, and the y coordinate of the centre is -3.
    (*a*) Let the centre be $C(x, -3)$ and write an equation stating that $CO = 5$.
    (*b*) Solve the equation for x.
    (*c*) Draw a sketch to illustrate your two answers.

6.(a) $(3, 8)$
7.(b) 3
8. $(x-2)^2 + (y+4)^2 = 25$
10.(a) $(3, -5)$
11.(a) $(x-2)^2 + (y-3)^2 = 52$
(b) $(x+3)^2 + (y-1)^2 = 5$
13.(b) $x = 1$
(d) $(x-1)^2 + y^2 = 5$

13. A circle passes through the points A$(-1, 1)$ and B$(0, -2)$ and has its centre on the x axis.
   (*a*) Let the centre have coordinates C$(x, 0)$ and write an equation showing that CA = CB.
   (*b*) Solve the equation for x.
   (*c*) Find the length of the radius.*
   (*d*) Find the equation of the circle.

Why 0?

*CA or CB

12.3 Inequations and Circles

A What values of x satisfy the inequation

$$x > 3?$$

x could be 3.1, 3.2, ..., 4.5, ...

What values of y satisfy the inequation

$$y > 2?$$

You know that the points (x, y) on a circle of radius "a" and centre the origin satisfy

$$x^2 + y^2 = a^2.$$

But what points (x, y) satisfy

$$x^2 + y^2 < a^2 \quad \text{or} \quad x^2 + y^2 > a^2?$$

Let's examine a typical case.

1. (*a*) Draw the circle $x^2 + y^2 = 64$ on graph paper.
   (*b*) Plot the following points $(3, 4)$, $(5, 6)$, $(-3, 8)$, $(8, 0)$, $(2, 7)$, and $(-5, 5)$ on the same graph.

2. (*a*) Copy and complete the following table.

Point	$x^2 + y^2$	$= 64$ > 64 < 64	Is the point (*a*) on the circle? (*b*) inside the circle? (*c*) outside the circle?
(x, y)	$x^2 + y^2$	$= 64$	on the circle
$(3, 4)$	$9 + 16$	< 64	inside the circle
$(5, 6)$			
$(-3, 8)$			
$(8, 0)$			
$(2, 7)$			
$(-5, 5)$			

   (*b*) What conclusions could you make regarding $x^2 + y^2$ compared to r^2 for points inside and outside the circle?

GREATER THAN

LESS THAN

GREATER THAN OR EQUAL TO

LESS THAN OR EQUAL TO

B Let's try to graph the relation defined by

$$\{(x, y) \mid x^2 + y^2 < 9\}.$$

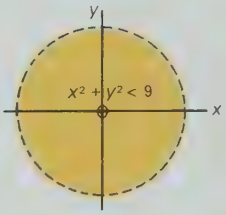
We know the points that satisfy

$$x^2 + y^2 = 9 \quad \text{or} \quad \sqrt{x^2 + y^2} = 3$$

is the set of points making up the circle with centre $(0, 0)$ and radius 3.

A broken line indicates that the boundary line is not included.

Now, $x^2 + y^2 < 9$ can be written as $\sqrt{x^2 + y^2} < 3$ and is satisfied by points (x, y) whose distance from $(0, 0)$ is less than 3, that is, all points *inside* the circle $x^2 + y^2 = 9$. The circle $x^2 + y^2 = 9$ is a boundary of the set of points we required and is not included in the set.



1. Graph each inequation.
   (*a*) $x^2 + y^2 \leqslant 9$
   (*b*) $x^2 + y^2 > 16$
   (*c*) $4x^2 + 4y^2 \geqslant 9$
   (*d*) $9x^2 + 9y^2 < 1$

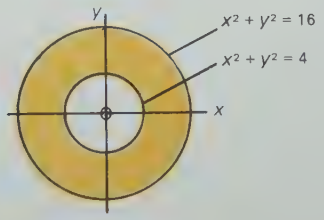
C We should also be able to find an inequation or several inequations that will describe a graph where regions are involved. For example, look at the following graphs.

(*a*)

Since the required points are in the first quadrant, $x > 0$ and $y > 0$. We have to include parts of the axes and the points on and inside the quarter circle.

$\therefore x^2 + y^2 \leqslant 9,\ x \geqslant 0,\ y \geqslant 0$ describes the graph.

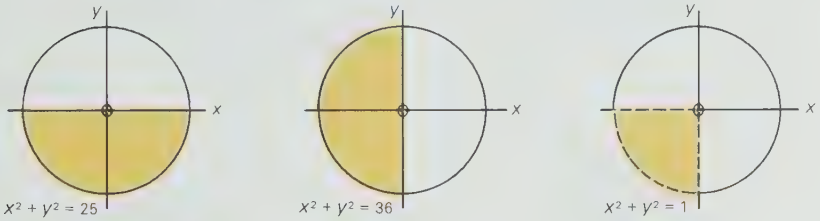
(*b*)



We are concerned with the points inside the circle $x^2 + y^2 = 16$ and outside the circle $x^2 + y^2 = 4$, plus the points on the boundary lines. The boundary lines are $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$.
$\therefore \{(x, y) | x^2 + y^2 \leqslant 16$ AND $x^2 + y^2 \geqslant 4\}$ describes the graph.

* Why not use "OR"?

1. Describe each graph using inequations.



Notice the broken line.

Exercise 12.3

Written Solutions

1. Determine whether the following points are interior to, exterior to, or on the circle defined by $x^2 + y^2 = 49$.
   (*a*) $(4, 5)$
   (*b*) $(7, 2)$
   (*c*) $(5, 6)$
   (*d*) $(-3, 8)$
   (*e*) $(5, 5)$
   (*f*) $(-2, 3\sqrt{5})$
   (*g*) $(-4, \sqrt{34})$
   (*h*) $(5, -2\sqrt{6})$
   (*i*) $(-\sqrt{40}, 3)$
   (*j*) $(-4, -2\sqrt{6})$

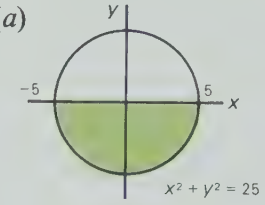
2. Draw graphs of the following relations.
   (*a*) $\{(x, y) | x^2 + y^2 \leqslant 16\}$
   (*b*) $\{(x, y) | x^2 + y^2 \geqslant 9\}$
   (*c*) $\{(x, y) | x^2 + y^2 < 25, x > 0, y > 0\}$
   (*d*) $\{(x, y) | x^2 + y^2 \leqslant 36, x < 0, y < 0\}$
   (*e*) $\{(x, y) | x^2 + y^2 < 4, x > 0, y \geqslant 1\}$

3. (*a*) Find an equation of the circle with centre at $(1, 2)$ and radius of length 4 units.
   (*b*) State an inequation that defines the relation for the set of points interior to the circle given in part (*a*).
   (*c*) State inequations that describe the set of points exterior to the circle in (*a*).

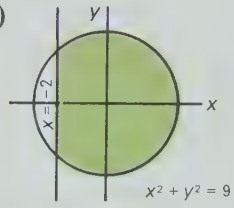
1.(a) interior (f) on
(d) exterior (i) on
2.(a) $\{(x, y) | x^2 + y^2 \leqslant 16\}$
Centre (0, 0) Radius = 4
(c) $\{(x, y) | x^2 + y^2 < 25,$
$x > 0, y > 0\}$
3.(a) $(x - 1)^2 + (y - 2)^2 = 16$

4. Use set builder notation to define the relation represented by each graph.

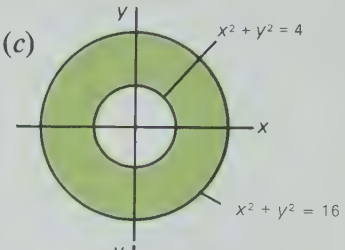
(a)



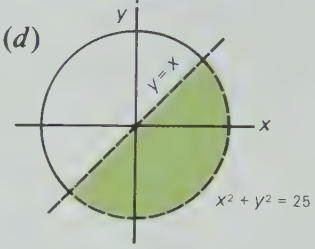
(b)



(c)



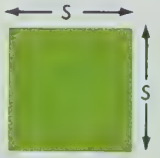
(d)



4.(a) $\{(x, y) | x^2 + y^2 \leq 25, y \leq 0\}$
(d) $\{(x, y) | x^2 + y^2 < 25, y < x\}$

12.4 The Quadratic Relation

We have already found out that equations like $y = mx + b$ give straight lines and are therefore known as Linear Relations. Now let us look at equations of Second Degree Relations, like $y = 2x^2, y = -5x^2, y = x^2 + 2$, and $y = -2x^2 + 5$.



1. Consider a square with sides s units in length. Let A represent the area of the square. Copy and complete the following table of values.

s	0	1	2	3	4	5
A						

2. Plot the ordered pairs from the table of values (s along the horizontal axis).
3. Would the graph be a series of line segments joining the five points or would it be a smooth curve?
4. To show that the graph is continuous and smooth, complete the following table and add the points to the graph.

s	0.25	0.5	0.75	1.25	1.5
A					

5. State an equation that relates A and s.

The curve you obtained is the partial graph of a quadratic relation of the form $y = ax^2$, $a \neq 0$.

6. Copy and complete the following tables of values.

$y = x^2$	
x	y
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	

$y = 2x^2$	
x	y
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	

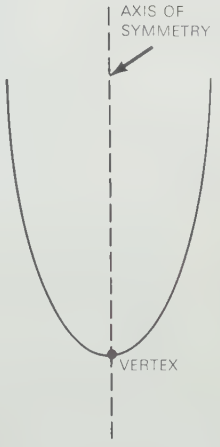
$y = \frac{1}{2}x^2$	
x	y
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	

Quadratic graphs of this form are called parabolas.

7. (*a*) Sketch the graphs of each of the relations in Question 6 on the same axes.
   (*b*) What is the smallest value of y in each graph?
   (*c*) At what value of x does this occur?

These values are the coordinates of the *minimum point* of the curve. This point is called the *vertex*.

   (*d*) State the domain and range for the relation.



8. Copy and complete the following table.

Relation	*Direction of Opening*
$y = x^2$	
$y = 2x^2$	
$y = \frac{1}{2}x^2$	

Graphs of quadratic relations of the form $y = ax^2$ are called *parabolas*. Note that the parabola is symmetrical about a line passing through the vertex. This line is called the *axis of symmetry*.

9. Copy and complete the following tables of values.

$y = -x^2$	
x	y
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	

$y = -2x^2$	
x	y
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	

$y = -\frac{1}{2}x^2$	
x	y
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	

10. (*a*) Sketch graphs of each of the relations in Question 9 on the same axes.
    (*b*) What is the largest value of y in each graph?
    (*c*) At what value of x does this occur?

These values are the coordinates of the *maximum point* of the curve. Again, this point is called the *vertex*.

    (*d*) State the domain and range for the relation.

11. Copy and complete the following table.

Relation	*Direction of Opening*
$y = -x^2$ $y = -2x^2$ $y = -\frac{1}{2}x^2$	

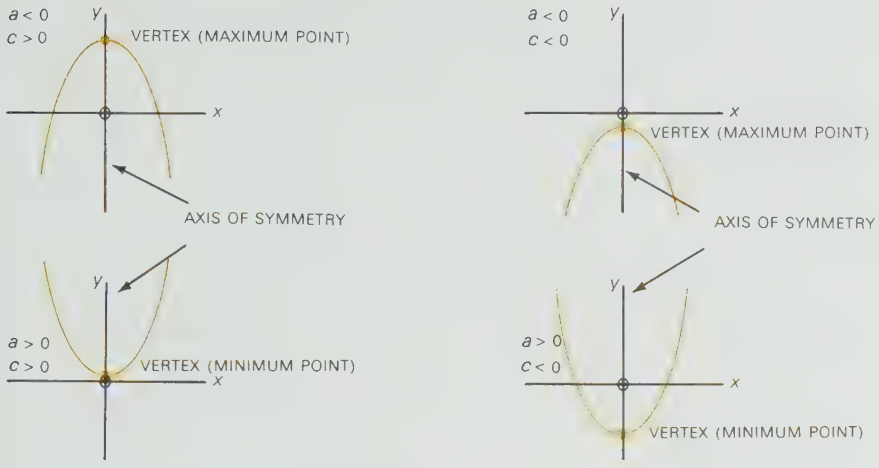
12. From the graphs of the quadratic relations, state the relationship between the positive and negative values of "a" and the direction of opening of the quadratic.

13. (*a*) Using the same set of axes and values of x from -4 to 4, sketch the graphs defined by $y = x^2 + 2$ and $y = x^2 - 2$.
    (*b*) How do the graphs in (*a*) and the graph of $y = x^2$ differ?

14. (*a*) Using the same set of axes and a domain $\{x \mid -6 \leqslant x \leqslant 6\}$, sketch the graphs defined by $y = -\frac{1}{2}x^2 + 3$ and $y = -\frac{1}{2}x^2 - 3$.
    (*b*) How do these graphs differ from $y = -\frac{1}{2}x^2$?

Summary

$$y = ax^2 + c$$



Exercise 12.4

Written Solutions

1. Copy and complete the following table and draw the graphs of the relations defined by each of the following.

(*a*) $y = x^2 + 3$
(*b*) $y = x^2 - 1$
(*c*) $y = 2x^2 + 3$
(*d*) $y = -2x^2 - 1$

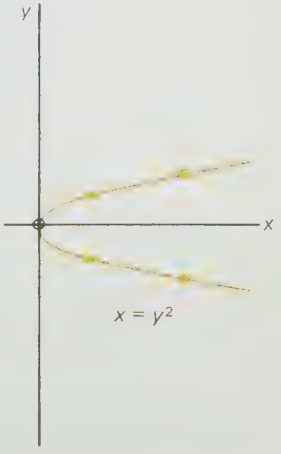
x	$x^2 + 3$	$x^2 - 1$	$2x^2 + 3$	$-2x^2 - 1$
-3				
-2				
-1				
0				
1				

2. State the coordinates of the maximum or minimum point of each graph in Question 1.

3. Copy and complete the following table and draw the graphs defined by each of the following.

(*a*) $x = y^2$
(*b*) $x = y^2 + 2$
(*c*) $x = 2y^2$
(*d*) $x = 2y^2 - 1$

y	y^2	$y^2 + 2$	$2y^2$	$2y^2 - 1$
-3				
-2				
-1				
0				
1				



2.(a) Minimum point = (0, 3)
(b) Minimum point = (0, −1)
(c) Minimum point = (0, 3)
(d) Maximum point = (0, −1)

4. Using the same set of axes, graph the relations given by $y = x^2 + c$ where $c = 1, 2, 0, -1$, and -2.
   (*a*) What type of curve is each graph?
   (*b*) State the coordinates of the maximum or minimum point of each curve (vertex).
   (*c*) How does the value of c affect the position of the parabola?

5. Using the same set of axes, draw the graphs of the parabolas $y = x^2 + 4$ and $y = -x^2 - 4$.
   (*a*) What is the direction of opening of each parabola?
   (*b*) What are the coordinates of the vertex of each curve?
   (*c*) State the domain and range for each parabola.

6. Given the equation of a parabola $y = x^2 + c$.
   (*a*) For what values of c does the parabola just touch the x axis?
   (*b*) For what values of c does the parabola intersect the x axis in two points?
   (*c*) For what values does the curve not intersect the x axis?

7. Given the equation of a parabola $y = -x^2 + c$.
   (*a*) For what values of c does the parabola just touch the x axis?
   (*b*) For what values of c does the parabola intersect the x axis in two points?
   (*c*) For what values does the parabola not meet the x axis?

8. Draw the graph of the parabola $y = x^2 - 4x + 1$
   (*a*) State the coordinates of the vertex.
   (*b*) Label the axis of symmetry.
   (*c*) State the domain and range.

9. Repeat Question 8 for the following.
   (*a*) $y = -x^2 - 4x - 5$
   (*b*) $y = x^2 + 6x + 10$
   (*c*) $y = -x^2 + 2x + 1$
   (*d*) $y = -x^2 - 2x - 5$
   (*e*) $y = x^2 - 4x + 5$

4.(b) (0, 1), (0, 2), (0, 0), (0, −1) (0, −2)

5.(a) $y = x^2 + 4$ **opens up;** $y = -x^2 - 4$ **opens down**

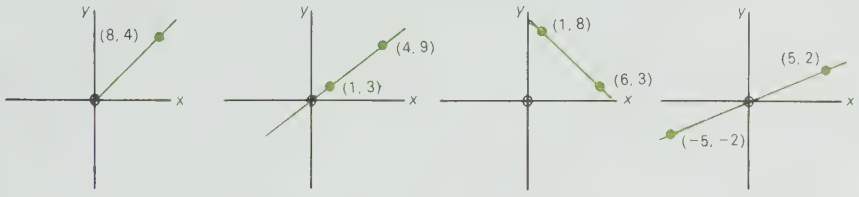
6.(a) $c = 0$
(b) $c < 0$
(c) $c > 0$

8.(a) 2, −3
(b) $x - 2 = 0$
(c) $\{x | x \in R\}$, $\{y | y \geq -3\}$

Maintaining Basic Skills 3

1. Find the slope of the line segments.

(*a*) (*b*) (*c*) (*d*)

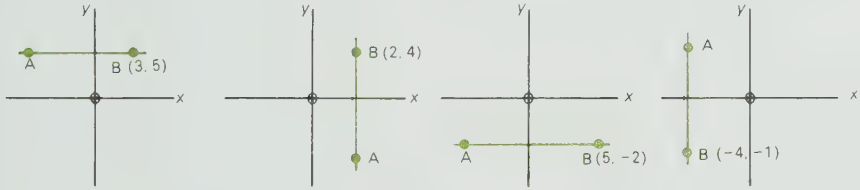


2. Find the slope of the line joining the given points.

(*a*)	(6, 5), (7, 8)	(*d*)	(3, 12), (7, 8)	(*g*)	(−2, 3), (4, −2)
(*b*)	(2, 8), (4, 0)	(*e*)	(5, 6), (8, 6)	(*h*)	(−3, 1), (−2, 5)
(*c*)	(4, 5), (6, 11)	(*f*)	(4, 3), (6, −5)	(*i*)	(6, 2), (−3, −4)

3. Find the equation of the line AB.

(*a*) (*b*) (*c*) (*d*)

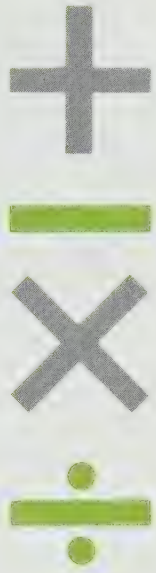


4. Find the equation of the following lines.
   (*a*) through (3, 6), parallel to the x axis
   (*b*) through (6, 2), parallel to the y axis
   (*c*) through (2, −3) parallel to the x axis
   (*d*) through (−2, −1) perpendicular to the x axis
   (*e*) through (0, 0) parallel to the x axis
   (*f*) through (−1, 7) perpendicular to the y axis
   (*g*) the y axis

5. Find the equation of the following lines.
   (*a*) through (0, 0) having a slope of 3
   (*b*) through (0, 0) with slope of −2
   (*c*) through (0, 0) with a slope of $\frac{4}{5}$
   (*d*) through the origin, with a slope of $-\frac{3}{7}$

6. Find the equation of the lines through the given point and having the given slope m.

(*a*)	(1, 3), $m = 2$	(*e*)	(5, 2), $m = \frac{2}{3}$
(*b*)	(2, 5), $m = -5$	(*f*)	(4, −3), $m = -\frac{3}{5}$
(*c*)	(6, −2), $m = 3$	(*g*)	(−1, −2), $m = -\frac{3}{4}$
(*d*)	(−4, 1), $m = -4$	(*h*)	(4, 1), $m = 0$



1.(a) $\frac{1}{2}$ **(b) 2**

2.(a) 3 **(c) 3**
(b) −4

3.(c) $y = -2$ **(d)** $x = -4$

4.(e) $y = 0$ **(g)** $x = 0$
(f) $y = 7$

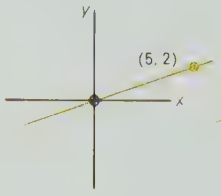
5.(a) $y = 3x$ **or** $3x - y = 0$
(b) $y = -2x$ **or** $2x + y = 0$

6.(c) $3x - y = 20$
(d) $4x + y = -15$
(g) $3x + 4y = -11$
(h) $y = 1$

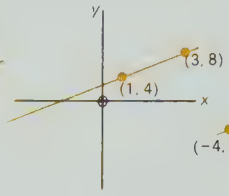
7. Find the equation through the given points.
   (*a*) (0, 0), (3, 5)
   (*b*) (0, 0), (−2, 4)
   (*c*) (6, 0), (0, 0)
   (*d*) (0, 0), (−5, 0)

8. Find the equation of each line.

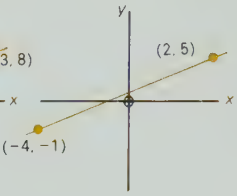
(*a*)



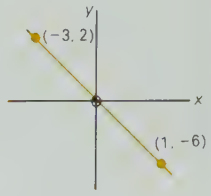
(*b*)



(*c*)



(*d*)



9. Find the equation of the line joining AB.
   (*a*) A(2, 6), B(4, 10)
   (*b*) A(3, 1), B(5, 7)
   (*c*) A(4, 3), B(2, 5)
   (*d*) A(0, 4), B(2, 6)
   (*e*) A(−2, 3), B(1, 5)
   (*f*) A(3, −1), B(5, 2)

10. State the slope of each line.
   (*a*) $y = 3x + 5$
   (*b*) $y = -4x + 8$
   (*c*) $2y = 7x - 3$
   (*d*) $4x + y = 12$
   (*e*) $x + 7y = 9$
   (*f*) $2x - 3y = 7$

11. Find the x-intercept, the y-intercept, and graph each line.
   (*a*) $x + y = 3$
   (*b*) $x - y = 4$
   (*c*) $4x - y = 12$
   (*d*) $2x + 5y = 20$
   (*e*) $x - 4y = 8$
   (*f*) $4y - 2x = 11$

12. Give the coordinates of the points where each line crosses the x axis and the y axis.
   (*a*) $3x - y = 4$
   (*b*) $2x - 3y = 12$
   (*c*) $4x + y = 8$
   (*d*) $5x + 3y - 30 = 0$
   (*e*) $x + 5 - 2y = 0$
   (*f*) $2x - 5y = 9$
   (*g*) $3x + 2y - 5 = 0$
   (*h*) $4y - 2x + 1 = 0$

13. Find the equation of the line joining P and Q.
   (*a*) P(6, 3), Q(8, 3)
   (*b*) P(−2, 1), Q(1, 1)
   (*c*) P(2, −5), Q(−1, −5)
   (*d*) P(4, 2), Q(−3, 2)

14. Find the equation of the following lines.
   (*a*) through (3, 7), having an x-intercept of 4
   (*b*) through (2, 5), having a y-intercept of 3
   (*c*) through (−3, 1), crossing the x axis at $x = -2$
   (*d*) through (−4, −2), crossing the y axis at $y = -3$

7.(c) $y = 0$ **(d)** $y = 0$

8.(a) $2x - 5y = 0$
(b) $2x - y = -2$

9.(b) $3x - y = 8$
(c) $x + y = 7$

10.(a) 3 **(d)** -4
(b) -4

11.(a) x-intercept = 3, y-intercept = 3
(c) x-intercept = 3, y-intercept = -12
(e) x-intercept = 8, y-intercept = -2

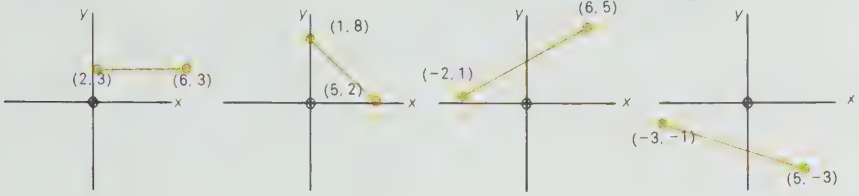
13.(c) $y = -5$ **(d)** $y = 2$

14.(a) $7x + y = 28$
(b) $x - y = -3$

15. Find the equation of the line having
   (*a*) a slope of 3, y-intercept of 5
   (*b*) a slope of 4, x-intercept of 7
   (*c*) a slope of -1, crossing the y axis at $y = 2$
   (*d*) a slope of $-\frac{2}{3}$, crossing the x axis at $x = 4$

16. State the coordinates of the midpoint of the line segments.

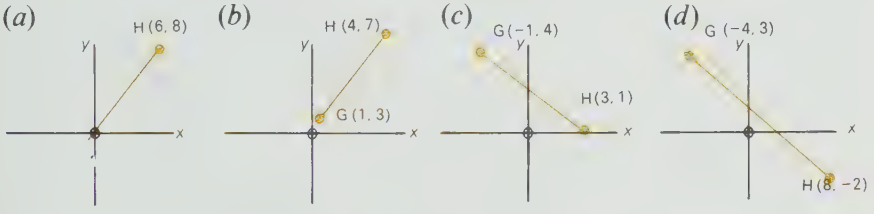
(*a*) (*b*) (*c*) (*d*)



17. Find the coordinates of the midpoint of the line segment joining these points.
   (*a*) (3, 1), (3, 5)
   (*b*) (−2, 3), (−2, −1)
   (*c*) (6, 5), (4, 5)
   (*d*) (−3, 2), (5, 4)
   (*e*) (3, 5), (1, 10)
   (*f*) (4, 3), (7, 6)
   (*g*) (2, 1), (−4, −3)
   (*h*) (7, −5), (8, −6)

18. Calculate the length of the line segment GH.

(*a*) (*b*) (*c*) (*d*)



19. Find the length of the line segment joining RS.
   (*a*) R(0, 0), S(0, 8)
   (*b*) R(0, 5), S(0, 12)
   (*c*) R(6, 0), S(−2, 0)
   (*d*) R(0, 0), S(3, 4)
   (*e*) R(−5, 12), S(0, 0)
   (*f*) R(6, 0), S(0, 8)
   (*g*) R(0, −5), S(12, 0)
   (*h*) R(0, 0), S(3, −2)

20. Find the distance between the given points.
   (*a*) (3, 1), (6, 5)
   (*b*) (2, 7), (−2, 3)
   (*c*) (3, 5), (8, −7)
   (*d*) (2, 3), (4, 6)
   (*e*) (1, −5), (−2, 1)
   (*f*) (1, −3), (4, −7)
   (*g*) (−15, 32), (−13, 27)
   (*h*) $(\frac{1}{2}, \frac{7}{4})$, $(\frac{5}{2}, \frac{11}{4})$

15.(c) $x + y = 2$
(d) $2x + 3y = 8$

16.(a) (4, 3) **(b)** (3, 5)

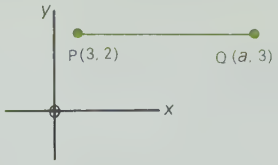
17.(a) (3, 3) **(c)** (5, 5)
(b) (−2, 1) **(d)** (1, 3)

18.(c) 5 **(d)** 13

19.(a) 8 **(c)** 8
(b) 7 **(d)** 5

20.(e) $3\sqrt{5}$ **(g)** $\sqrt{29}$
(f) 5 **(h)** $\sqrt{5}$

21. (*a*) PQ has a length of 6 units. Find a.



(*b*) The length of the line segment RQ is 10, where R$(-8, b)$, Q$(0, 0)$. Find b.

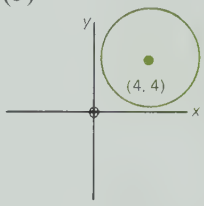
(*c*) The distance from T$(a, 5)$ to the origin is 13. Find a.

(*d*) The distance between S$(6, 8)$ and W$(6, y)$ is 7. Find y.

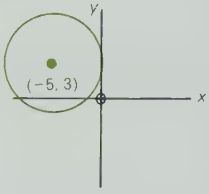
22. Give the equation of the circles shown.

(*a*)

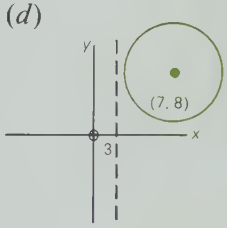
(*b*)



(*c*)



(*d*)



23. State the radius and centre of each circle.

(*a*) $x^2 + y^2 = 9$

(*b*) $x^2 + y^2 = 3$

(*c*) $(x - 2)^2 + (y - 4)^2 = 25$

(*d*) $(x + 3)^2 + (y - 1)^2 = 9$

(*e*) $(x - 1)^2 + (y + 2)^2 = 3$

(*f*) $(x + 5)^2 + (y + 3)^2 = 7$

24. Give the equation of the following circles.

(*a*) radius 3, centre (2, 5)

(*b*) centre $(-3, 8)$, radius 4

(*c*) centre $(-4, -3)$, diameter 2

(*d*) centre $(8, -4)$, radius $\sqrt{5}$

25. For each of the parabolas given, (i) make a table of values for $x = -3, -2, -1, 0, 1, 2, 3$, (ii) graph the ordered pairs obtained, (iii) state the vertex, (iv) give the equation of the axis of symmetry.

(*a*) $y = 2x^2$

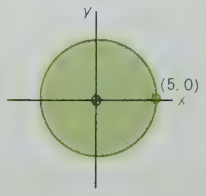
(*b*) $y = x^2 - 2$

(*c*) $y = x^2 + 4$

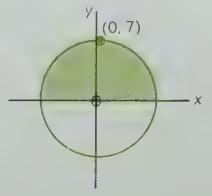
(*d*) $y = \frac{1}{2}x^2 - 3$

26. Describe the shaded region.

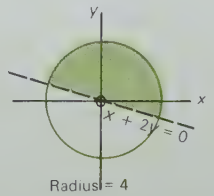
(*a*)



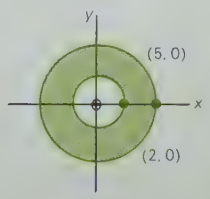
(*b*)



(*c*)



(*d*)



21.(a) $a = 8$ **(b)** $b = \pm 6$

22.(a) $x^2 + y^2 = 25$
(b) $x^2 + y^2 - 8x - 8y + 16 = 0$

24.(a) $(x - 2)^2 + (y - 5)^2 = 9$ **or** $x^2 + y^2 - 4x - 10y + 20 = 0$

25.(a) (iii) Vertex : (0, 0)
(c) (iii) Vertex : (0, 4)

26.(a) $\{(x, y) | x^2 + y^2 \leq 25\}$

27. Given that a varies directly as b. When $a = 14$, $b = 2$.
   (*a*) Find the equation relating a and b.
   (*b*) When $b = 4$, what is the value of a?

28. Given that c is proportional to h. When $h = 4$, $c = 20$.
   (*a*) Find the equation relating c and h.
   (*b*) When $h = 3$, what is the value of c?

29. When $m = 10$, $n = 2$, and m varies inversely as n.
   (*a*) What is the equation describing the variation?
   (*b*) Find the value of m when $n = 5$.

30. A is partly constant and partly varies as B. When $A = 11$, $B = 2$ and when $A = 17$, $B = 4$.
   (*a*) Find the equation.
   (*b*) If $B = 5$, what is the value of A?

31. $x \propto y$. When $x = 15$, $y = 5$.
   (*a*) Find the equation and the proportionality constant.
   (*b*) When $y = 4$, what is the value of x?
   (*c*) When $y = 8$, what is the value of x?
   (*d*) If $x = 21$, what is the value of y?

32. $r \propto s$. $r = 9$ corresponds to $s = 18$.
   (*a*) Find the equation and the proportionality constant.
   (*b*) Find r when $s = 20$.
   (*c*) Find r when $s = 8$.
   (*d*) Find s when $r = 7$.

33. $x \propto y^2$. When $y = 2$ m, $x = 20$ m^2.
   (*a*) Give the proportionality constant and the equation.
   (*b*) Find x when $y = 3$ m.
   (*c*) If $y = \sqrt{7}$ m, what is the value of x?
   (*d*) Find x when $y = 0.4$ m.

34. The time for one complete swing of a simple pendulum, T, varies directly as the square root of its length, l. When $l = 36$ cm, $T = 1.2$ s.
   (*a*) Find the equation relating T to l.
   (*b*) When $l = 25$ cm, how long does one complete swing take?
   (*c*) What is the time for one complete swing for a pendulum whose length is 16 cm?
   (*d*) If the time for one complete swing is 1.5 s, what is the length of the pendulum, to the nearest centimetre?

27.(a) $a = 7b$
(b) $a = 28$

29.(a) $mn = 20$
(b) $m = 4$

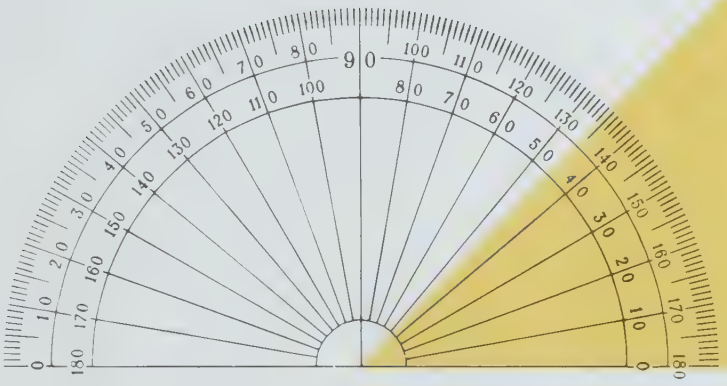
31.(c) $x = 24$
(d) $y = 7$

32.(c) $r = 4$
(d) $s = 14$

33.(a) $k = 5$, $x = 5y^2$
(b) $x = 45$ m^2

34.(c) 0.8 s
(d) 56 cm

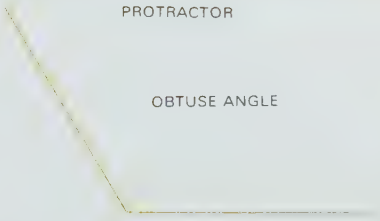
Unit 13
Angles



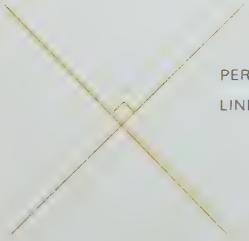
PROTRACTOR

RIGHT ANGLE

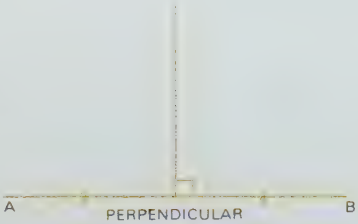
ACUTE ANGLE



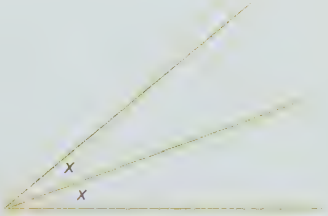
OBTUSE ANGLE



PERPENDICULAR
LINES



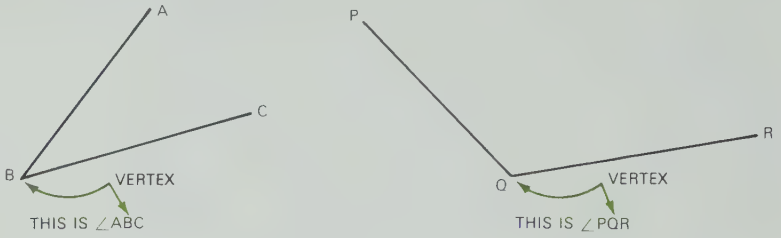
PERPENDICULAR
BISECTOR OF AB



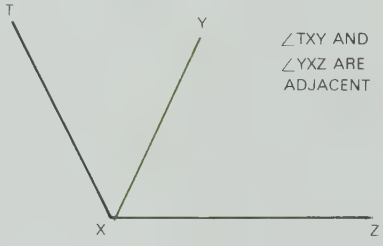
EQUAL ANGLES

13.1 Angle Review

1. Remember how angles are named?

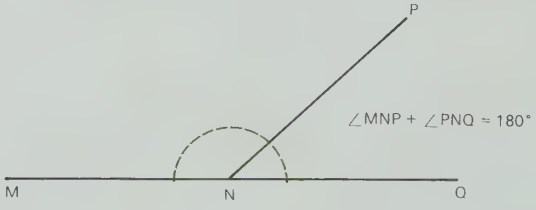


2. *Adjacent angles* share a common arm and vertex and lie on opposite sides.



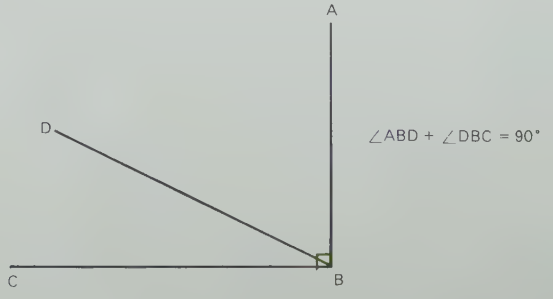
3. If adjacent angles form a straight line (180°), they are called *supplementary angles.*

Supplementary angles (180°)

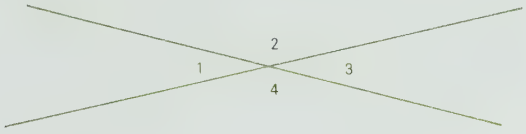


4. If adjacent angles form a right angle (90°), they are called *complementary angles.*

Complementary angles (90°)



5. When two straight lines intersect, four angles are formed. Angles 1 and 3 are *opposite angles*. Angles 2 and 4 are, too.



It *looks* as though $\angle 1$ and $\angle 3$ are equal, and $\angle 2$ and $\angle 4$ are equal. But can we *prove* it?

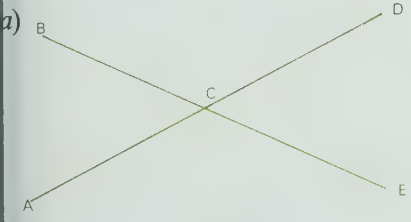
Since $\angle 1$ and $\angle 2$ are supplementary angles,
$\angle 1 + \angle 2 = 180$ (degrees).
Since $\angle 2$ and $\angle 3$ are also supplementary angles,
$\angle 2 + \angle 3 = 180$ (degrees).
We could say $\angle 1 = 180 - \angle 2$
and $\angle 3 = 180 - \angle 2$
$\therefore \angle 1 = \angle 3$
Similarly, $\angle 2 = \angle 4$.

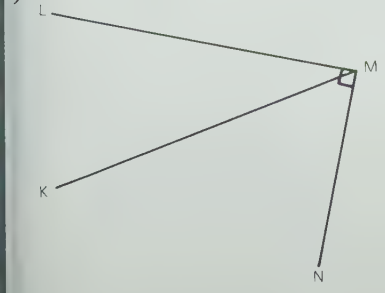
Proved!

Exercise 13.1

Discussion

1. Use the diagrams to find examples of adjacent angles, supplementary angles, complementary angles, and opposite angles.

(a)


(b)


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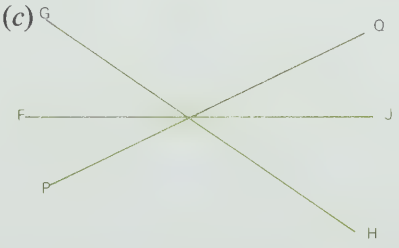
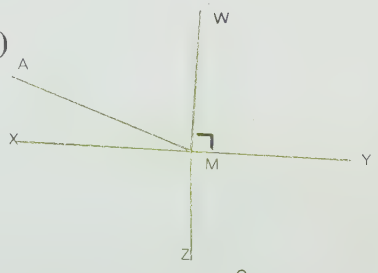


(d)

(e)


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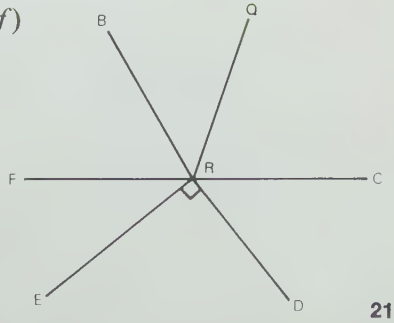
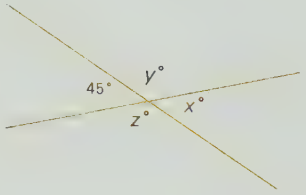


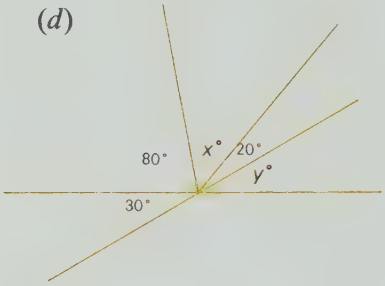
Written Solutions

2. Determine the value of the variables.

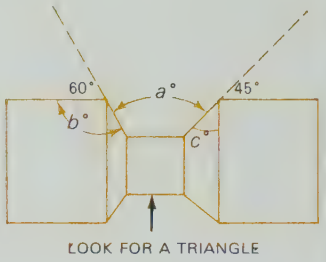
(a)



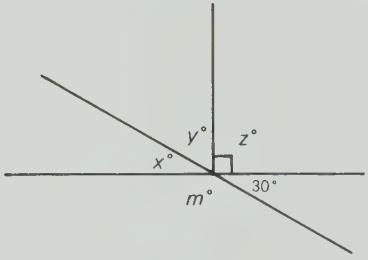
(d)



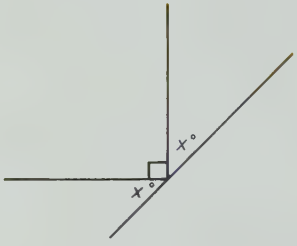
(g)



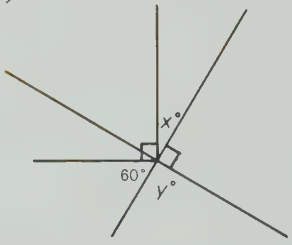
(b)



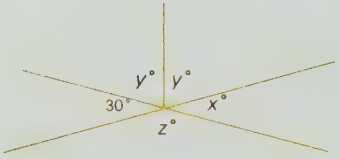
(e)



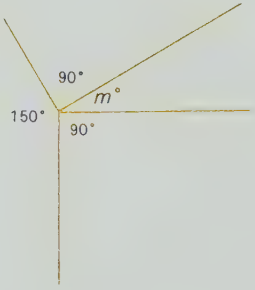
(h)



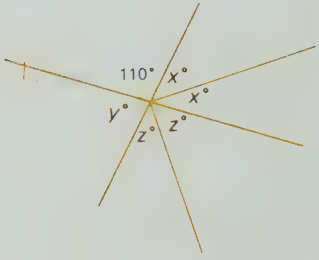
(c)

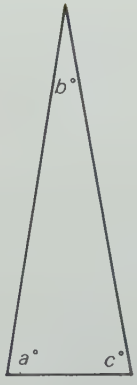


(f)



(i)

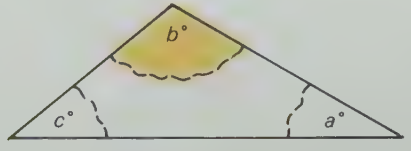




It works with any triangle.
Try it!

13.2 Angles in a Triangle

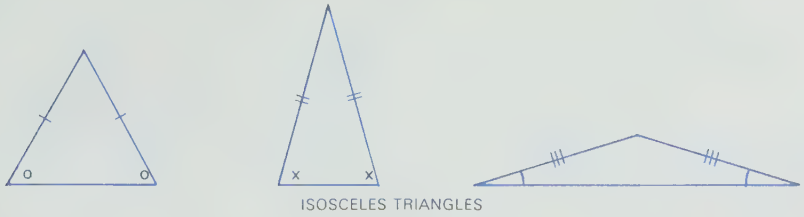
A If you have never tried this before, draw *any* triangle on paper, tear off the corners, and fit them together.



$\therefore\ a + b + c = 180$ (DEGREES)

B Recall what you have learned about these two special triangles.

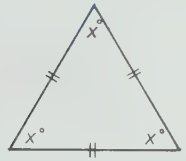
(*a*)



ISOSCELES TRIANGLES

What parts of an isosceles triangle are equal?

(*b*)



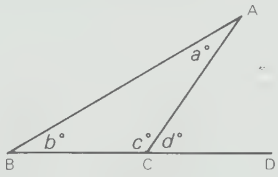
EQUILATERAL TRIANGLE

Since all angles are equal,

$$x + x + x = 180$$
$$3x = 180$$
$$x = 60$$

C If any one side of a triangle is extended, the angle formed is called an *exterior angle*.

We know that $a + b + c = 180$
And we know that $c + d = 180$
$\therefore a + b + c = c + d$
and $a + b = d$

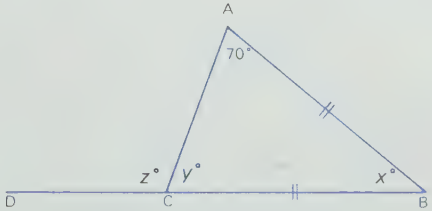


$\angle ACD$ is an exterior angle.

Notice that angles a and b are the two interior opposite angles to the exterior angle d.

Example

Calculate the value of the variables.



Solution

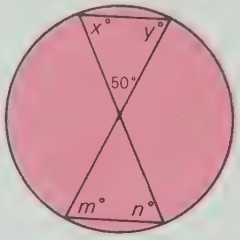
$y = 70$ ← Isosceles $\triangle$
$x = 40$ ← $\triangle$ sum 180
$z = 110$ ← $\angle A + \angle B$ (or $180 - y$)

Exercise 13.2

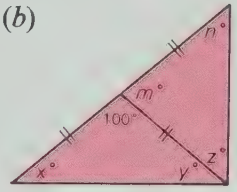
Discussion

1. Find the value of each variable.

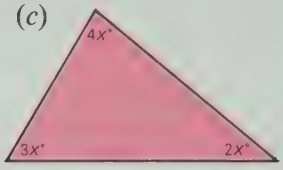
(a)



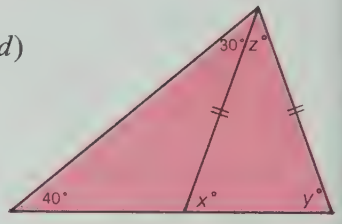
(b)



(c)



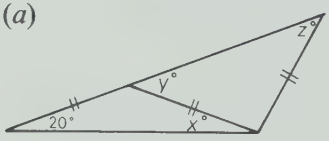
(d)



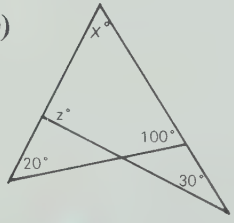
Written Solutions

2. Find the value of each variable.

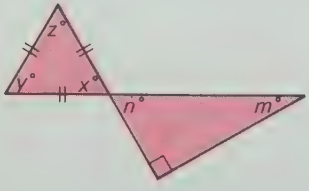
(a)



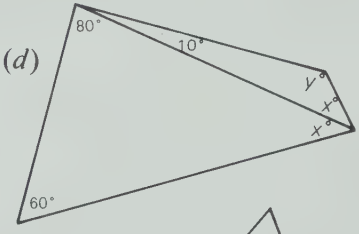
(b)



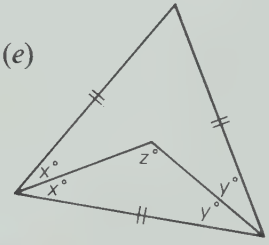
(c)



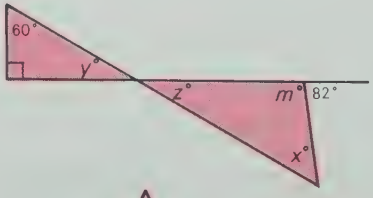
(d)



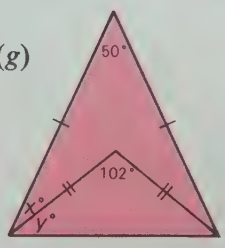
(e)



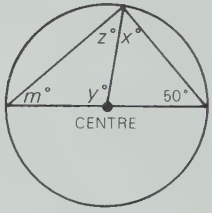
(f)



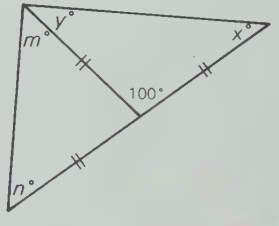
(g)



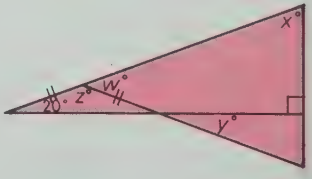
(h)



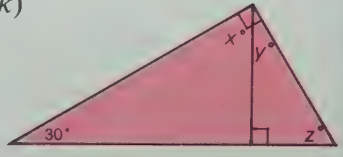
(i)



(j)



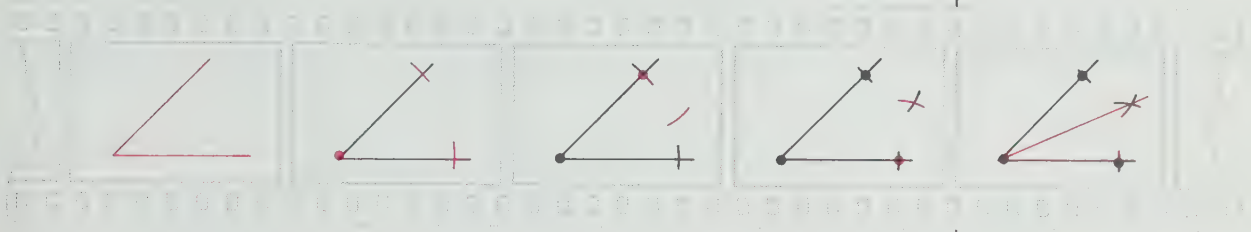
(k)



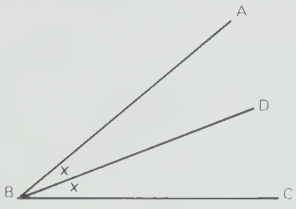
2.(a) $x^\circ = 20^\circ$ **(c)** $x^\circ = 60^\circ$, $m^\circ = 30^\circ$ **(e)** $y^\circ = 30^\circ$ **(f)** $x^\circ = 52^\circ$, $m^\circ = 98^\circ$ **(g)** $x^\circ = 26^\circ$ **(h)** $x^\circ = 50^\circ$ **(i)** $x^\circ = 40^\circ$, $n^\circ = 50^\circ$ **(j)** $y^\circ = 20^\circ$ **(k)** $x^\circ = 60^\circ$

13.3 Circles and Triangles

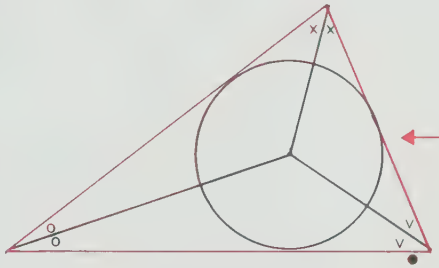
A Remember how to bisect an angle?



When an angle is bisected, the adjacent angles formed are equal. Here $\angle ABD = \angle DBC$.

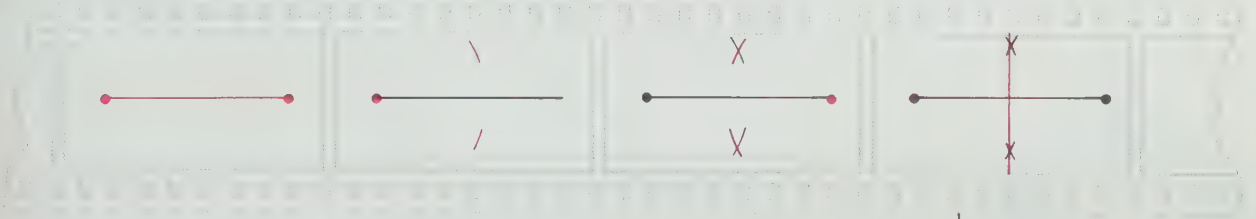


It is an interesting fact that the bisectors of the three angles of any triangle all meet at a common point. This point is the centre of the largest circle that is contained entirely within the triangle. Test this with several triangles of different shapes.

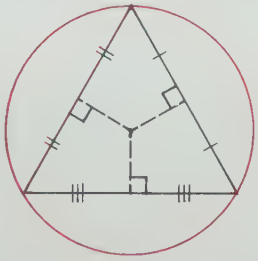


The inscribed circle of the triangle.

B Remember how to draw a perpendicular bisector?



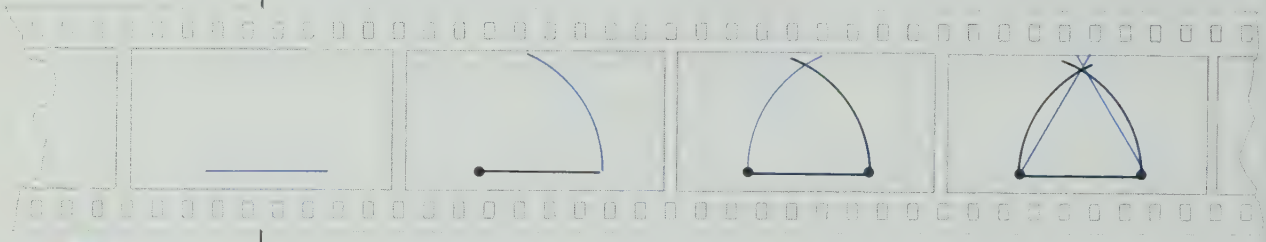
The perpendicular bisectors of the sides of a triangle also meet at a common point. This point is the centre of the circle that passes through all three vertices of the triangle. Try this construction with several triangles.



The circumscribed circle of the triangle.

Exercise 13.3

1. Draw any angle and use the method suggested in the filmstrip to bisect the angle.
2. Draw any line segment and draw its perpendicular bisector using the method suggested in the filmstrip.
3. Draw an angle of 180°. Using this 180° angle and a set of compasses, make angles of 90°, 45°, $22\frac{1}{2}$°.
4. (*a*) Draw an equilateral triangle using the method suggested.



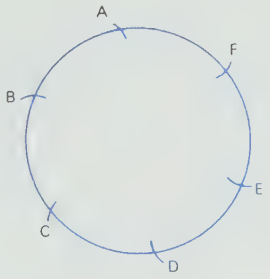
(*b*) What is the size of each angle of an equilateral triangle?

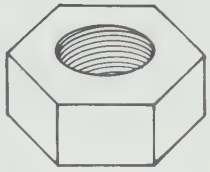
5. Construct an angle of 60°. Bisecting this angle, make angles of 30° and then 15°. How could you make a 75° angle?
6. Using only ruler and compasses, construct a triangle having angles of 60°, 45°, and 75°. Inscribe a circle in this triangle.
7. Construct a triangle having angles 30°, 60°, 90°. Inscribe a circle in the triangle.
8. Construct a triangle having angles of 15°, 60°, 105°. Inscribe a circle in the triangle.
9. Draw the circumscribed circle for a triangle having angles of 45°, 30°, 105°.
10. Draw the circumscribed circle for a triangle having angles of 60°, 30°, 90°.
11. Did you notice anything special about the centre of the circle drawn in Question 10? Check your observation by drawing any other right-angled triangle and circumscribing a circle.

You've got me cornered!

Now it's my turn!

12. (*a*) Draw any circle and use the circle radius to find the points A, B, C, D, E, and F on it.
(*b*) Join A to B, B to C, etc.
(*c*) What kind of figure is ABCDEF?
(*d*) Why is a hexagonal shape used for nuts that are used on bolts?
(*e*) What shape is used by bees in making the inside structure of their hives?





What are you? Some kind of nut?

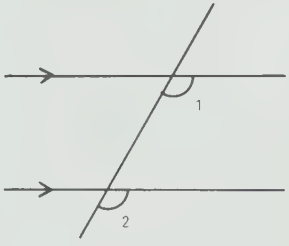
13.4 Angles and Parallel Lines

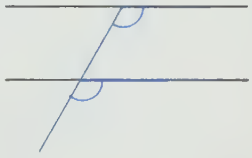
In previous work we discovered the following facts concerning angles formed when a line (transversal) intersects two parallel lines.

(*a*) *F* pattern

Corresponding angles are equal.

$\angle 1 = \angle 2$

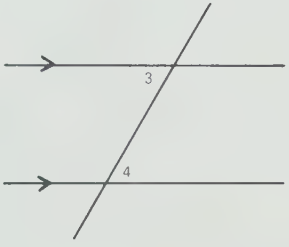




(*b*) *Z* pattern

Alternate angles are equal.

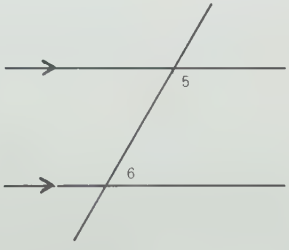
$\angle 3 = \angle 4$

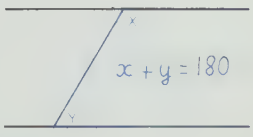


(*c*) *C* pattern

Interior angles on the same side of the transversal are supplementary.

$\angle 5 + \angle 6 = 180$

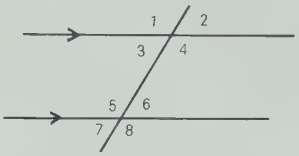




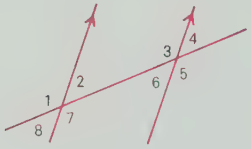
Exercise 13.4

Discussion

1. Name all the pairs of corresponding angles in the diagram.



2. Name all the pairs of alternate angles in the diagram.

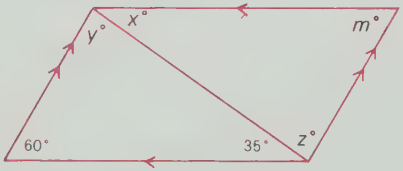


3. In the diagram of Question 2 name the pairs of interior angles on the same side of the transversal. What is true about them?

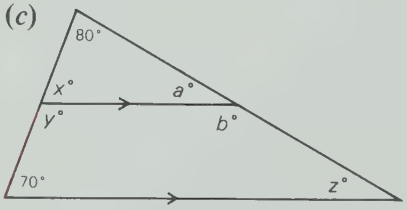
Written Solutions

4. Determine the value of the variables indicated in the diagrams.

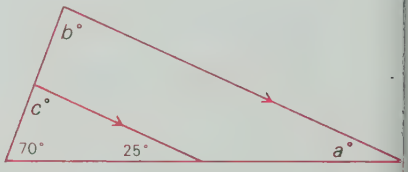
(a)



(c)



(e)



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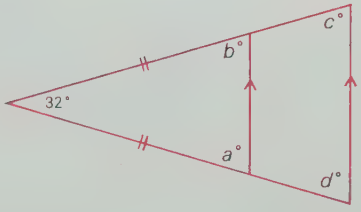
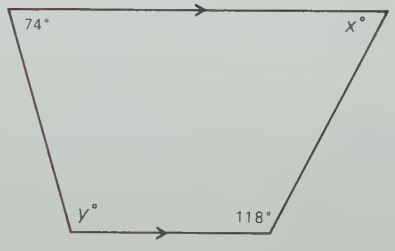
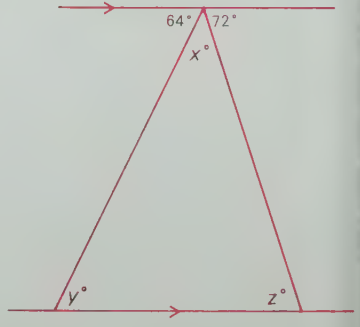


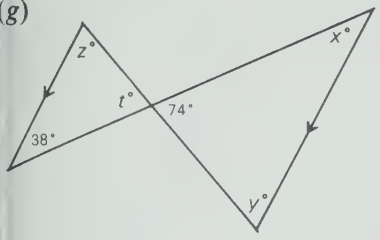
(d)



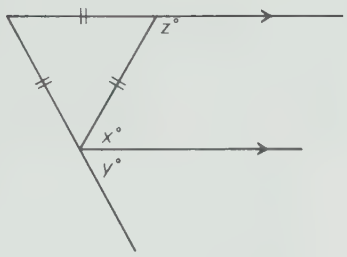
(f)



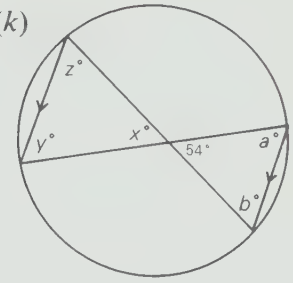
(g)



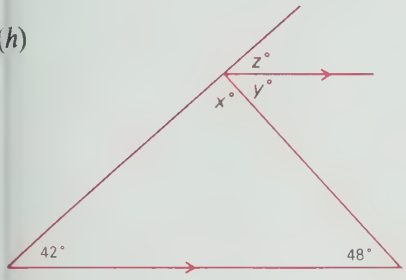
(i)



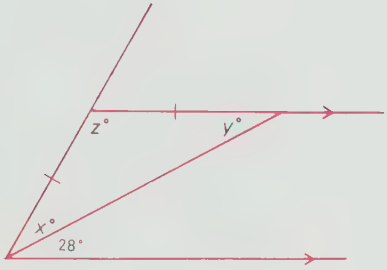
(k)



(h)



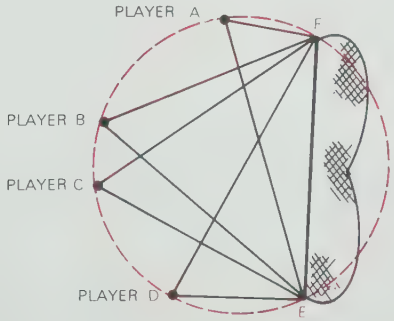
(j)



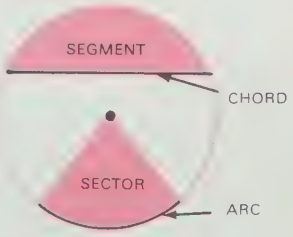
4.(a) $x^\circ = 35^\circ$
$y^\circ = 85^\circ$
(b) $a^\circ = 74^\circ$
$b^\circ = 74^\circ$
(c) $x^\circ = 70^\circ$
$y^\circ = 110^\circ$
(d) $x^\circ = 62^\circ$
(e) $a^\circ = 25^\circ$
(f) $x^\circ = 44^\circ$
(g) $x^\circ = 38^\circ$
$t^\circ = 74^\circ$
(h) $x^\circ = 90^\circ$
(i) $x^\circ = 60^\circ$
(j) $x^\circ = 28^\circ$
(k) $x^\circ = 54^\circ$
$b^\circ = 63^\circ$

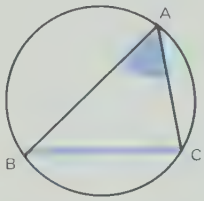
13.5 Angles in a Circle

A The diagram shows a number of hockey players standing on the circumference of a circle of which the goal line is a chord. Which player has the largest shooting angle? Measure and see!

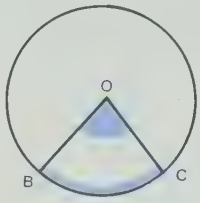


The four angles A, B, C, D all "stand" on the same side of chord FE and have their vertices on the circumference of the circle. All angles like this will be found to be equal. They are called *angles in the same segment*.





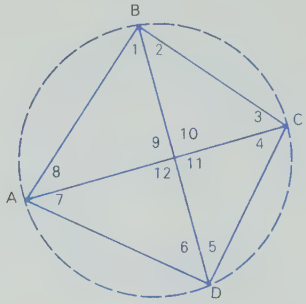
∠A is an angle at the circumference standing on chord BC
OR
angle in segment BAC
OR
segment angle BAC.



∠BOC is the angle at the centre standing on arc BC
OR
sector angle BOC.

Example

Name all the equal angles in the given figure.

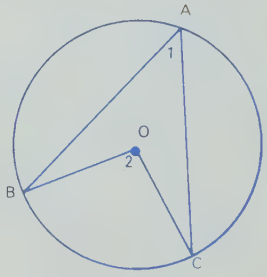


Solution

$\angle 1 = \angle 4$ (same segment)
$\angle 8 = \angle 5$ (same segment)
$\angle 10 = \angle 12$ (opposite angles)
$\angle 9 = \angle 11$ (opposite angles)
$\angle 3 = \angle 6$ (same segment)
$\angle 2 = \angle 7$ (same segment)

B Another interesting fact involves the angles marked 1 and 2 in this diagram.

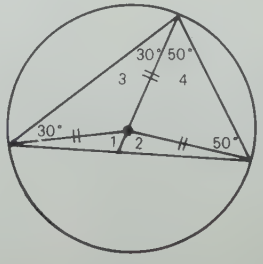
$\angle$BAC is an angle at the circumference and angle BOC is an angle at the centre and both "stand" on chord BC. Measure the two angles and draw a conclusion about them. Is this fact true about any such related angles? Try some and see!



It can be proven that the angle at the centre will always be double the angle at the circumference as long as they are on the same side of the same chord. Test this idea with the diagram.

Find $\angle 1 + \angle 2$.

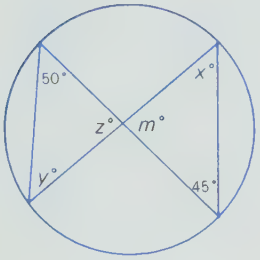
Show that
$\angle 1 + \angle 2 = 2(\angle 3 + \angle 4)$.



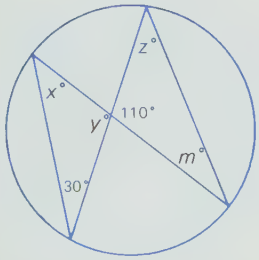
Exercise 13.5

Calculate the value of each variable.

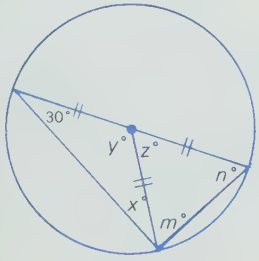
1.



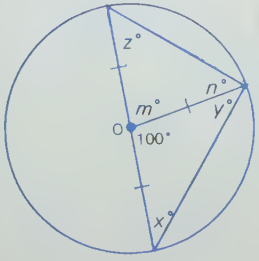
2.



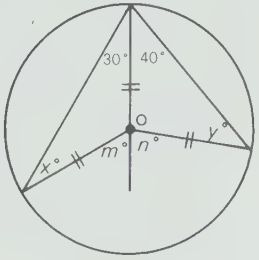
3. Compare $z°$ to $30°$ and $y°$ to $n°$.



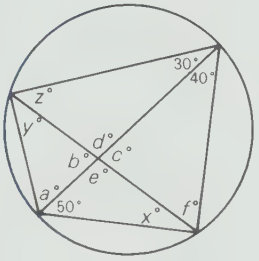
4. Compare $z°$ to $100°$ and $x°$ to $m°$.



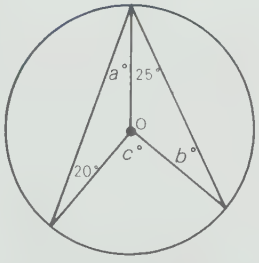
5. Compare $m° + n°$ to $30° + 40°$.



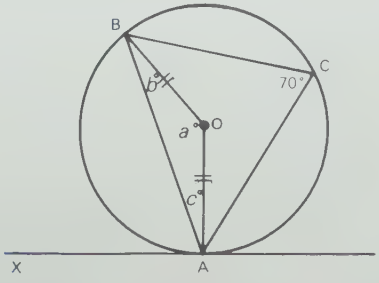
6.



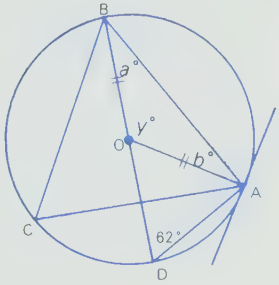
7.



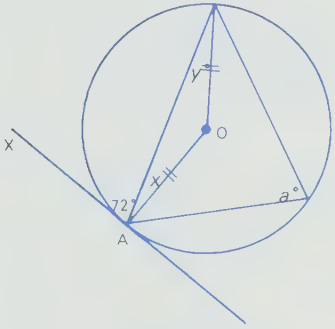
8. (*a*) Measure angle XAO.
(*b*) Calculate angle XAB.



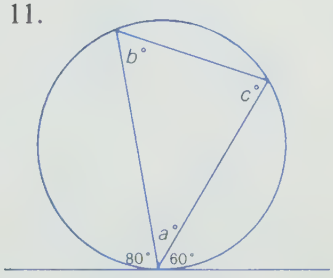
9. (*a*) Measure angle XAO.
(*b*) Calculate angle XAB.



10. Compare $72°$ to $a°$.

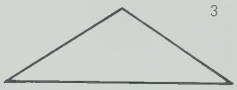


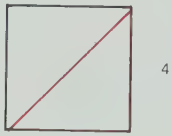
11.



1. $x° = 50°$, $y° = 45°$
2. $z° = 40°$, $m° = 30°$
3. $z° = 60°$, $m° = 60°$
4. $m° = 80°$, $n° = 50°$
5. $x° = 30°$, $y° = 40°$
6. $a° = 60°$, $b° = 80°$
7. $b° = 25°$
8. $b° = 20°$, $c° = 20°$
9. $b° = 28°$
10. $y° = 18°$
11. $a° = 40°$

13.6 Angles in a Polygon





Copy and complete the table shown.

Number of sides	*Number of diagonals from one vertex*	*Number of triangles formed*	*Sum of interior angles*
3	0	1	180°
4	1	2	360°
5			
6			
7			
8			

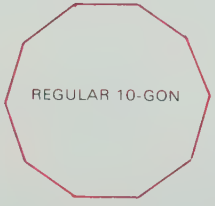
What is the relationship between the number of sides of the polygon and the number of non-overlapping triangles that can be formed within it?

If a polygon has n sides, the number of non-overlapping triangles would be $n - 2$. Therefore the sum of the interior angles would be $(n - 2)\ 180°$.

This fact can be used to calculate the size of each angle in a regular polygon since all angles in a regular polygon are equal.

Example

Calculate the size of each angle in a regular 10-gon.



Solution

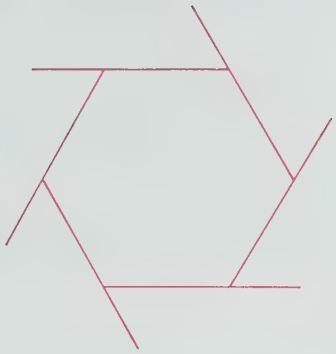
The total of all 10 angles is $(10 - 2)\ 180° = 8 \times 180°$
$= 1440°$

Therefore each angle is $\frac{1440°}{10} = 144°$.

Exercise 13.6

1. Calculate the size of each angle in these regular figures.
   (*a*) pentagon
   (*b*) hexagon
   (*c*) octagon
   (*d*) 12-gon
   (*e*) 20-gon

2. In the diagram all sides of the regular hexagon have been extended in the same manner to form exterior angles. Calculate the measure of each exterior angle. Calculate the total of these exterior angles.



3. Repeat Question 2 for a regular octagon and a regular 10-gon. What conclusion can you draw about the sum of the exterior angles of a polygon?
4. A hexagon has two angles of $x°$, two angles of $3x°$ and two angles of $5x°$. Calculate the measure of each of these angles.
5. How many sides has a regular polygon that has an exterior angle of 24°?
6. Prove that it is impossible to have a regular polygon with an exterior angle of 16°.

1.(a) 108°
(d) 150°

3. 45°; 360°
36°; 360°

5. 15

Remember that the exterior angles total 360°.

Unit 14
Measures of Angles

14.1 Measurement of Rotation

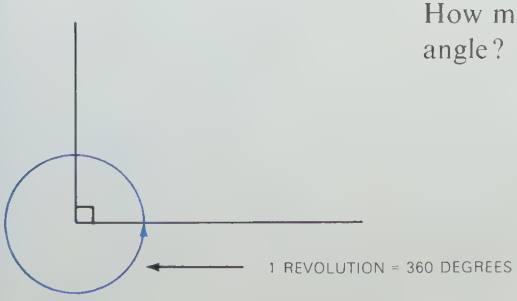
In today's mechanical world we are constantly turning things. We turn a door handle to open a door, turn a tap to obtain some water, turn the dial on our TV set to get a certain program, and so on.

In practice it is often necessary for people like engineers, surveyors, and navigators to measure angles. The unit they use is called a degree and it is equal to $\frac{1}{360}$th of a complete turn.

Early astronomers calculated that the earth revolves around the sun approximately once in every 360 days. Therefore in one day the earth revolves 1 degree or $\frac{1}{360}$th of a complete revolution.

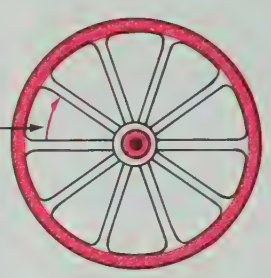
When is a car not a car?
When it turns into a driveway!

How many degrees in a right angle?



1. What is the angle between adjacent spokes of a wheel that has 10 spokes?

$\frac{1}{10}$ of 1 revolution $= \frac{1}{10} \times 360° = 36°$



2. What fraction of a revolution is 210°?
1 revolution is 360°.
$\therefore \frac{210°}{360°} = \frac{7}{12}$ of a revolution

3. Measure the angle between the two lines in the figure.

Remember the protractor?

Exercise 14.1

Written Solutions

1. The dial of a bathroom scale is in the form of a circle and is graduated in kilograms from 0 to 150. The pointer, in turning from the 0 division to the 150 division, turns through an angle of 360°. Through what angle does the pointer turn for persons with each of the following masses?
   (*a*) 100 kg
   (*b*) 20 kg
   (*c*) 5 kg
   (*d*) 120 kg
   (*e*) 60 kg
   (*f*) your mass

2. What will be the mass of a person if the pointer turns through each of the following angles?
   (*a*) 60°
   (*b*) 20°
   (*c*) 10°
   (*d*) 300°
   (*e*) 240°
   (*f*) 120°

1.(a) 240° (e) 144° (c) 12°

2.(a) 25 kg (e) 100 kg (d) 125 kg

3. The speedometer pointer in a car turns through a right angle when the speed of the car increases from 50 km/h to 100 km/h. Through what angle will the pointer turn when the car makes the following changes in speed?

(*a*) 0 km/h to 20 km/h
(*b*) 120 km/h to 80 km/h
(*c*) 20 km/h to 60 km/h
(*d*) 14 km/h to 19 km/h
(*e*) 38 km/h to 28 km/h
(*f*) 30 km/h to 55 km/h

4. Through what angle does the hour hand of a clock turn in each of the following time intervals?

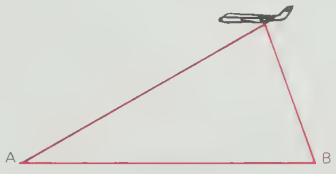
(*a*) 1 h
(*b*) 6 h
(*c*) 11 h
(*d*) 7 h
(*e*) 20 min
(*f*) 90 min
(*g*) 36 min
(*h*) 45 min

5. By estimation, draw any two lines making angles as follows.

(*a*) 90°
(*b*) 30°
(*c*) 45°
(*d*) 80°
(*e*) 60°
(*f*) 120°
(*g*) 300°
(*h*) 270°
(*i*) 240°

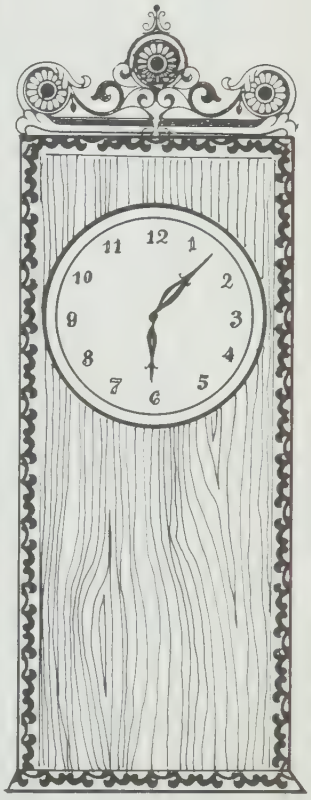
6. By checking with a protractor, compare your answers in Question 5 with the correct measure.

7. The position of a jet airliner could be found by finding its direction from two radio stations. If the distance between two stations A and B is known, and the angles A and B are found by radio operators, then a scale diagram can be drawn to find the airliner's position.



(*a*) Suppose AB = 80 km, $\angle A = 30°$ and $\angle B = 70°$. Using a scale of 1 cm = 10 km draw an accurate scale diagram and find
(i) the distance of the plane from A,
(ii) the distance of the plane from B.

(*b*) If after 10 min the plane has moved so that $\angle A = 120°$ and $\angle B = 15°$, find the new position of the plane on the same diagram. How many kilometres has the plane travelled in this time? How fast is it travelling?

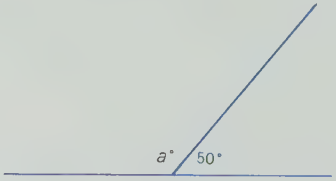


3.(a) 36° (e) 18°
4.(a) 30° (f) 45°
(d) 210° (g) 18°

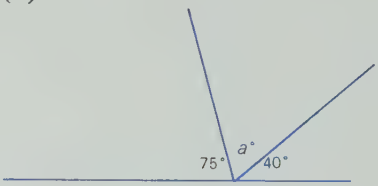
8.(a) 130°
(b) 60°
(c) 80°
9.(b) {5°, 85°}, {50°, 40°}

8. State the value of each variable.

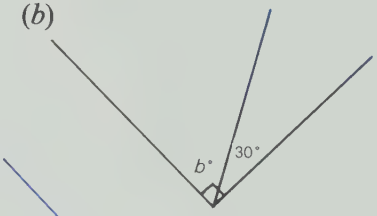
(*a*)



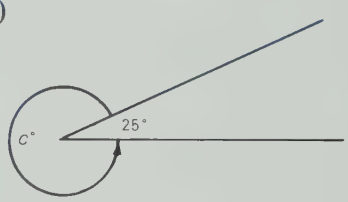
(*d*)



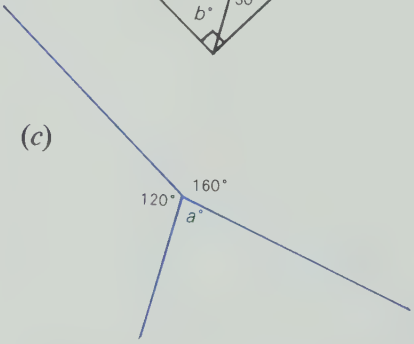
(*b*)



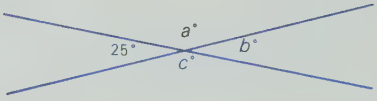
(*e*)



(*c*)



(*f*)



9. A = {5°, 50°, 70°, 140°, 95°}
B = {110°, 40°, 175°, 85°, 130°}
(*a*) Pair members of A with members of B so that together they make up an angle equalling a half-turn.
(*b*) Pair members that are equivalent to a right angle.

14.2 Units Smaller Than Degrees

A A measure of distance such as metres is subdivided into the smaller units of centimetres and millimetres. The degree measure of angles can be subdivided into minutes and seconds.

1 degree = 60 minutes or $1° = 60'$
1 minute = 60 seconds or $1' = 60''$

$30° \ 25' \ 22''$ is the measure of an angle and is read as 30 degrees, 25 minutes, and 22 seconds.

1. How many minutes in each of the following?

(*a*) 2°	(*c*) 30.5°	(*e*) 100°
(*b*) 60°	(*d*) 45°	(*f*) 90°

2. How many seconds in each of the following?

(*a*) 2′	(*c*) 10°	(*e*) 15°
(*b*) 10′	(*d*) 15′	(*f*) 1°

B We often must express fractions of degrees in minutes and seconds. If we want to express 30.56° in degrees, minutes, and seconds, we must follow two steps.

Step 1 Convert the decimal of the degrees part to minutes.

$$\therefore (0.56)^\circ = (0.56 \times 60)'$$
$$= 33.6'$$

Step 2 Convert the decimal of the minutes part to seconds.

$$\therefore (0.6)' = (0.6 \times 60)''$$
$$= 36''$$
$$\therefore 30.56^\circ = 30^\circ 33' 36''$$

Now try these.

Convert the following to degrees, minutes, and seconds.

(*a*) 24.24°	(*c*) 120.88°
(*b*) 12.26°	(*d*) 93.17°

Exercise 14.2

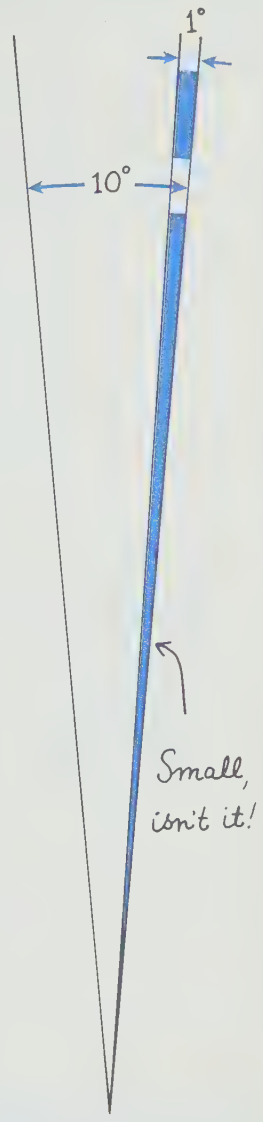
Written Solutions

1. Convert each of the following to minutes.

(*a*) 0.5°	(*f*) 0.132°
(*b*) 0.32°	(*g*) 0.921°
(*c*) 1.24°	(*h*) 1.567°
(*d*) 0.256°	(*i*) 10.241°
(*e*) 3.812°	(*j*) 21.817°

2. Convert each of the following to seconds.

(*a*) 0.25′	(*e*) 0.52′
(*b*) 0.33′	(*f*) 1.37′
(*c*) 2.45′	(*g*) 2.41′
(*d*) 0.80′	(*h*) 10.15′



$1^\circ = 60'$

$1' = 60''$

1.(a) 30′ (f) 7.92′
(c) 74.4′ (h) 94.02′

2.(a) 15″ (e) 31.2″
(d) 48″ (h) 609″

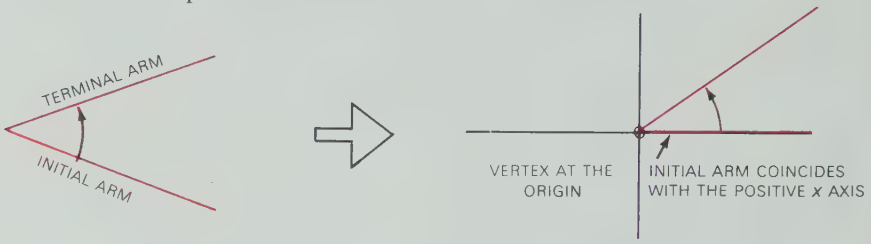
3.(a) 25° 24′ 36″
(c) 121° 27′ 36″
(e) 94° 33′ 39.6″
(f) 322° 25′ 44.4″
(h) 243° 37′ 25.8″
(j) 129° 59′ 20.4″

3. Convert each of the following to degrees, minutes, and seconds.

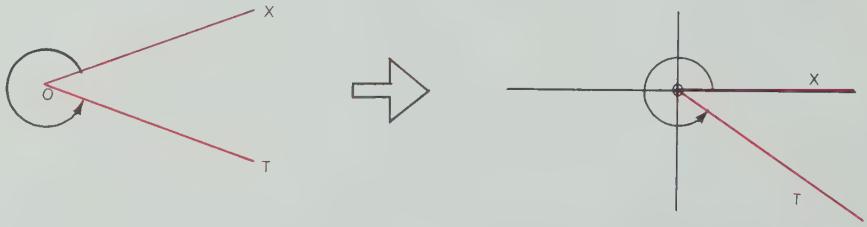
(*a*) 25.41°	(*f*) 322.429°
(*b*) 47.28°	(*g*) 194.815°
(*c*) 121.46°	(*h*) 243.623°
(*d*) 38.214°	(*i*) 148.788°
(*e*) 94.561°	(*j*) 129.989°

14.3 Angles in Standard Position

A An angle is said to be in *standard position* when its vertex is at the origin of a rectangular coordinate system and its initial arm coincides with the positive x axis.

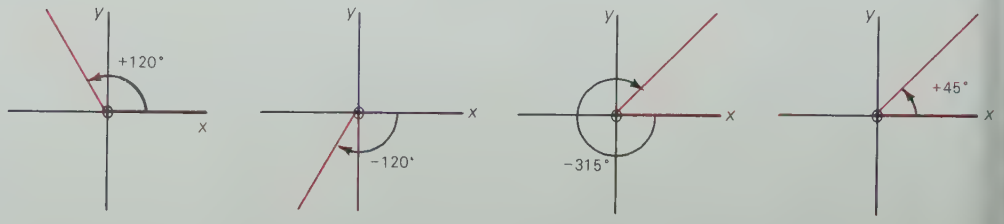


Angle is in standard position.

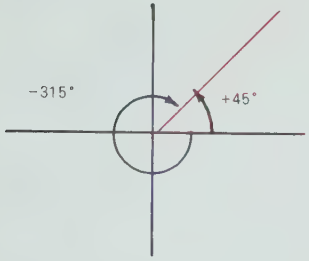


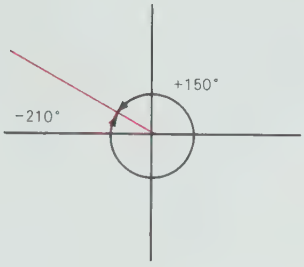
Angle XOT in standard position.

Angles of rotation may be taken either in the counter-clockwise direction or the clockwise direction. To make things easier, all angles formed by rotation in a counter-clockwise direction are considered positive angles, and angles formed by rotation in a clockwise direction are called negative angles.



B Angles that have the same terminal arms are called *coterminal angles.*



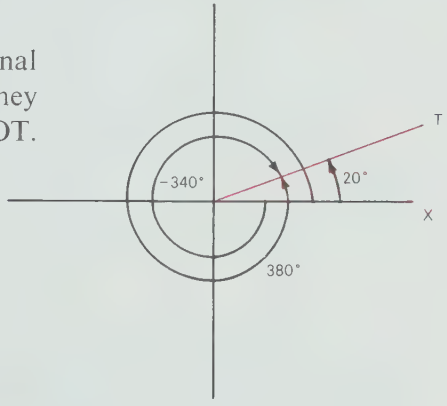


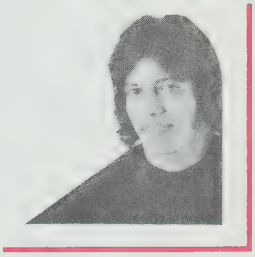
Example

Name at least two angles that are coterminal with a 20° angle and make a sketch showing all of these angles in standard position.

Solution

380°, −340° are coterminal angles with a 20° angle since they have the same terminal arm OT.





Exercise 14.3

Written Solutions

1. Draw each of the following as standard angles. Label each angle clearly.

(*a*) 135°	(*d*) −750°	(*g*) −10°
(*b*) 210°	(*e*) 1000°	(*h*) 345°
(*c*) 70°	(*f*) −460°	(*i*) −120°

2. State the measurement of the smallest positive angle coterminal with each angle in Question 1.

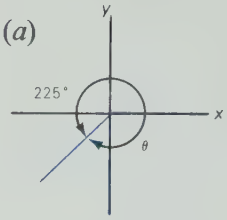
3. Which of the following angles are coterminal with an angle of 30°, if all are in standard position?

(*a*) 390°	(*d*) −330°	(*g*) 190°	(*j*) 10°
(*b*) 210°	(*e*) 750°	(*h*) −690°	(*k*) 370°
(*c*) 150°	(*f*) −30°	(*i*) 1110°	(*l*) 100°

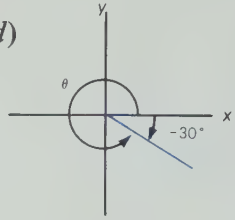
2.(a) 495°
(b) 570°
(c) 430°
(g) 350°
(h) 705°
(i) 240°

4. State the value of angle θ in each of the following diagrams.

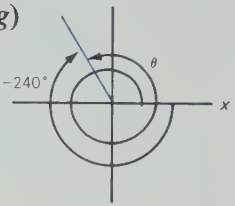
(a)



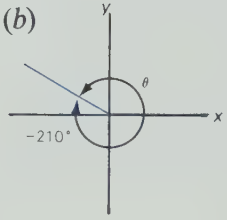
(d)



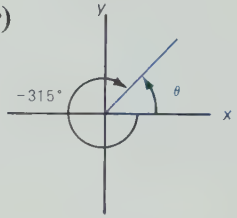
(g)



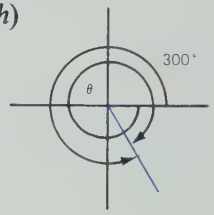
(b)



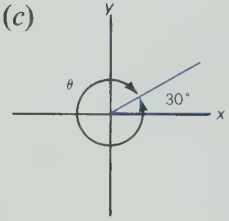
(e)



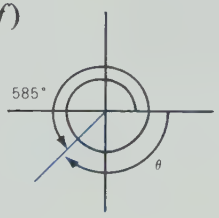
(h)



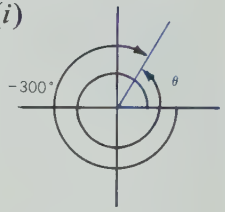
(c)



(f)



(i)



5. Which of the following angles are coterminal with an angle of $-100°$? All are in standard position.
   (a) 80° (c) 260° (e) $-460°$
   (b) 100° (d) 620°

6. State the measurement of the smallest negative angle coterminal with each of the following.
   (a) 40° (c) $-40°$ (e) 300°
   (b) 120° (d) $-100°$ (f) 360°

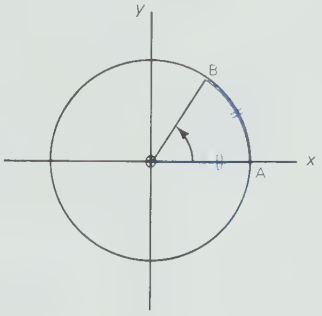
7. Sketch the face of a clock, marking 5 min intervals on it. Find the smallest positive angle formed by the two hands when the clock shows each of the following times.
   (a) 11:20 (c) 04:45 (e) 09:10
   (b) 02:30 (d) 10:15 (f) 06:05

4.(a) 135° (g) 480°
(d) 330°
6.(a) −320° (e) −60°
(c) −400°
7.(a) 140° (e) 145°
(c) 127.5°

14.4 The Radian Measure of an Angle

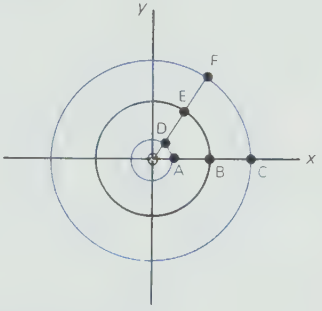
A Besides being measured in degrees, angles may also be measured in *radians*. This unit of angle measurement is used extensively in theoretical mathematics, and it is useful also in the study of trigonometry.

A circle is drawn on the Cartesian plane with centre at the origin. The angle AOB is subtended by an arc AB whose length is equal to the radius OA.



Arc AB = radius OA
$\therefore \angle AOB = 1$ radian (1 rad)
The measure of angle AOB is 1 rad.

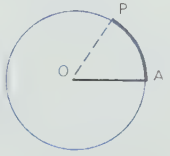
Regardless of the size of the circle, the radian is always the same size.
If arc AD = OA, arc BE = OB, arc CF = OC, then $\angle AOD$, $\angle BOE$, and $\angle COF$ all have a measure of 1 rad (1 rad $\doteq$ 57°).



> 1 rad is the angle subtended at the centre of a circle by an arc equal in length to the radius.

What is the radian measure of an angle subtended at the centre of a circle by an arc *not* equal in length to a radius?

If arc AP = 1 radius, $\angle AOP$ = 1 rad
If arc AP = 2 radii, $\angle AOP$ = 2 rad
If arc AP = 3 radii, $\angle AOP$ = ▭ rad
If arc AP = $\frac{1}{2}$ radius, $\angle AOP$ = ▭ rad
If arc AP = x radii, $\angle AOP$ = ▭ rad



$$\therefore \text{number of radians} = \frac{\text{arc length}}{\text{1 radius}}.$$

Check each of the examples above to be sure this formula works.

rad is the symbol for radians.

A radian is an amount of turning.

"Subtended by" means "standing on".

∠AOB is subtended by arc AB.

Example 1

If the radius of a circle is 15 cm, find the angle at the centre if the arc length is 20 cm.

Solution Let x rad be the required angle.

$$\text{Number of radians} = \frac{\text{arc length}}{\text{radius}}$$

$$\therefore x = \frac{20}{15} = 1\frac{1}{3}$$

The angle at the centre is $1\frac{1}{3}$ rad.

rad is one of the supplementary units in SI.

Example 2

If the radius of a circle is 10 m, how far does one walk along the circle to turn through an angle of 2 rad?

Solution

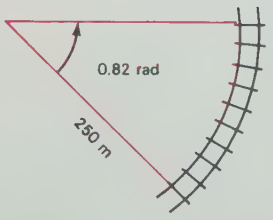
$$\text{Number of radians} = 2 = \frac{\text{arc length}}{\text{radius}}$$

$$\therefore 2 = \frac{\text{arc length}}{10}$$

$$\therefore \quad \text{arc length} = 20$$

$\therefore$ one walks 20 m in turning through 2 rad when the circle has a radius of 10 m.

1. In a circle of diameter 4 m, a central angle is 0.7 rad. Find the arc subtended by this angle.
2. A curved section of railroad track is an arc of a circle of radius 250 m. Find the length of the track if the radian measure of the central angle is 0.82 rad.

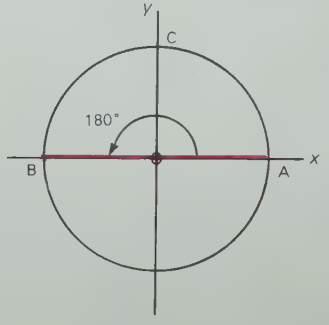


B It is often necessary to change from radians to degrees, and from degrees to radians.

To find the radian measure of an angle of 180°, draw a circle of any radius on the Cartesian plane.

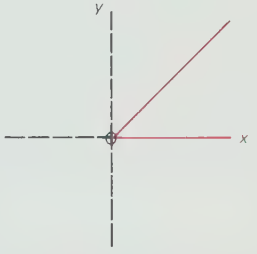
Draw an angle of 180°, in standard position. Call it $\angle AOB$.

$\angle AOB$ in radians $= \dfrac{\text{arc ACB}}{r}$



Since arc ACB is a semicircle, its length would be half the circumference of the circle. ($C = 2\pi r$)

$$\text{Arc ACB} = \tfrac{1}{2}(2\pi r)$$
$$= \pi r$$
$$\therefore \angle\text{AOB in radians} = \frac{\pi r}{r}$$
$$= \pi$$
$$\therefore \angle\text{AOB} = \pi \text{ rad}$$
But $\angle\text{AOB} = 180°$
$$\therefore \pi \text{ rad} = 180°$$



Remember standard angles?

π rad = 180°

Example 1

Express 135° in radian measure.

Solution

$$\text{Since } 180° = \pi \text{ rad}$$
$$1° = \frac{\pi}{180} \text{ rad}$$
$$135° = \frac{\pi}{180} \times 135 \text{ rad}$$
$$= \frac{3}{4}\pi \text{ rad}$$

Example 2

Express $\frac{\pi}{2}$ radians in degrees.

Solution

$$\text{Since } \pi \text{ rad} = 180°$$
$$\therefore \frac{\pi}{2} \text{ rad} = \frac{180°}{2}$$
$$= 90°$$

1. Change the following to radians.
   (*a*) 30° (*c*) 360° (*e*) 270°
   (*b*) 45° (*d*) 450° (*f*) 225°
2. Change the following to degrees.
   (*a*) $\frac{\pi}{3}$ rad (*c*) $\frac{2}{3}\pi$ rad (*e*) $\frac{7}{2}\pi$ rad
   (*b*) $\frac{\pi}{4}$ rad (*d*) $\frac{5}{4}\pi$ rad (*f*) 3π rad

Number of radians = arc / radius

60' = 1°
60" = 1'

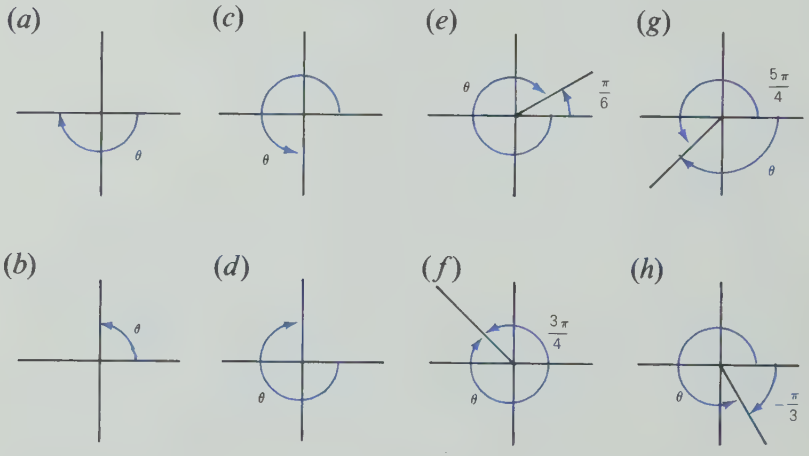
Exercise 14.4

Written Solutions

1. The radius of a circle is 3.5 cm. Find the number of radians in a central angle that is subtended by an arc of 2.75 cm.
2. An angle at the centre of a circle is 0.8 rad and is subtended by an arc of 14 cm. Find the radius of the circle.
3. Express 1 rad in degrees, minutes, and seconds. ($\pi = 3.14159$)
4. Express the following angles in radian measure (in terms of π).

(*a*) 180°	(*f*) 360°	(*k*) 90°
(*b*) 45°	(*g*) 210°	(*l*) 300°
(*c*) 30°	(*h*) 225°	(*m*) 720°
(*d*) 105°	(*i*) 240°	(*n*) 540°
(*e*) 18°	(*j*) 270°	(*o*) 0°

5. State the value of θ in radian measure for each angle.



6. Sketch the following angles in standard position.

(*a*) $\frac{3}{4}\pi$ rad	(*d*) $\frac{5}{6}\pi$ rad	(*g*) $\frac{11}{6}\pi$ rad
(*b*) $\frac{7}{6}\pi$ rad	(*e*) $\frac{4}{3}\pi$ rad	(*h*) $-\frac{3}{2}\pi$ rad
(*c*) $-\frac{2}{3}\pi$ rad	(*f*) $-\frac{9}{4}\pi$ rad	(*i*) $-\frac{8}{3}\pi$ rad

2. 17.5 cm

4.(a) π rad (i) $\frac{4}{3}\pi$ rad
(d) $\frac{7}{12}\pi$ rad (k) $\frac{\pi}{2}$ rad
(f) 2π rad (n) 3π rad

5.(a) $-\pi$ rad (g) $-\frac{3}{4}\pi$ rad
(b) $\frac{\pi}{2}$ rad (h) $\frac{5}{3}\pi$ rad

7. State the degree measure of each angle.

(a) $\frac{\pi}{4}$ rad (e) $\frac{3}{4}\pi$ rad (i) -2π rad

(b) $\frac{\pi}{3}$ rad (f) $\frac{5}{6}\pi$ rad (j) $-\frac{13}{6}\pi$ rad

(c) $\frac{\pi}{6}$ rad (g) $\frac{2}{3}\pi$ rad (k) 4π rad

(d) $\frac{\pi}{2}$ rad (h) $\frac{5}{3}\pi$ rad (l) $-\frac{9}{2}\pi$ rad

8. For each of the following angles in standard position, state the smallest positive coterminal angle (in radians).

(a) π rad (d) $-\frac{2}{3}\pi$ rad (g) $\frac{2}{3}\pi$ rad

(b) $\frac{\pi}{6}$ rad (e) $\frac{3}{4}\pi$ rad (h) $\frac{5}{4}\pi$ rad

(c) $-\frac{\pi}{2}$ rad (f) $-\frac{4}{3}\pi$ rad (i) $-\frac{3}{2}\pi$ rad

9. Two angles in standard position have the same terminal arm. If the measure of one of the angles is $\frac{\pi}{4}$, state three possible measures of the other angle (in radians).

10. Two angles are in standard position and have the same terminal arm. If the measure of one of the angles is $\frac{\pi}{6}$, state the value of the other angle θ if the following is true.

(a) $-2\pi \leqslant \theta \leqslant 2\pi$ (b) $2\pi \leqslant \theta \leqslant 4\pi$

11. Find the radian measure of the angle subtended by an arc of 1.6 m at the centre of the circle whose radius is 8 m.

12. An angle at the centre of a circle whose radius is 2.5 m is subtended by an arc of 7.5 m. What is the angle measurement in radians?

13. An arc of 18.43 m subtends an angle of 1.9 rad at the centre of the circle. Find the radius of the circle in centimetres.

14. The large hand of a clock is 63 cm long. How many centimetres does the tip move in 20 min? (Use $\pi = \frac{22}{7}$ and answer to nearest centimetre.)

Coterminal angles have the same terminal arm.

7.(a) $45°$ **(b)** $60°$ **(c)** $30°$ **(d)** $90°$ **(j)** $-390°$ **(k)** $720°$

8.(a) 3π rad **(g)** $-\frac{4}{3}\pi$ rad **(d)** $\frac{4}{3}\pi$ rad

10.(a) $-\frac{11}{6}\pi$ rad

12. 3 rad

14. 132 cm

Must be Paul Bunyan's pocket watch.

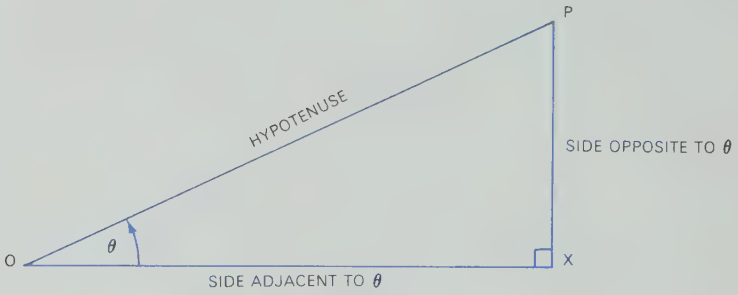
Unit 15
Trigonometry

The word trigonometry in its simplest sense means the measurement of triangles and the relations that exist between the sides, angles, and areas of triangles. Practical trigonometry is used by surveyors and navigators, but the use of analytic trigonometry has a much wider scope and involves all kinds of geometrical and algebraic investigations for use in electronics, physics, chemistry, and engineering.

15.1 Trigonometric Values

The trigonometric values of an angle are defined to be the values determined by ratios between the sides of a right triangle.

sine, *cosine*, *tangent* are usually shortened to *sin*, *cos*, *tan*



The value $\frac{XP}{OP}$ is called the *sine* of angle XOP.

The value $\frac{OX}{OP}$ is called the *cosine* of angle XOP.

The value $\frac{XP}{OX}$ is called the *tangent* of angle XOP.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Memory Device

SOH CAH TOA

S — sin
O — opp
H — hyp

C — cos
A — adj
H — hyp

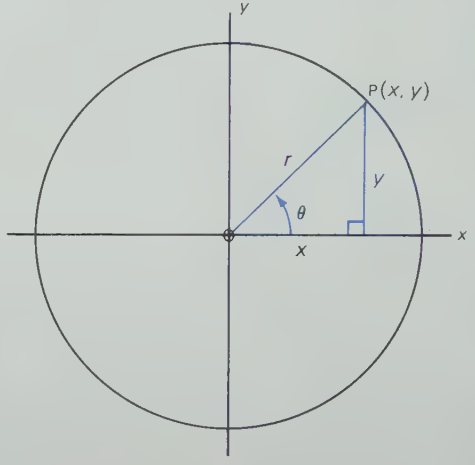
T — tan
O — opp
A — adj

If the triangle is placed on the Cartesian plane with the angle θ in standard position, we can define the ratios in terms of the coordinates of a point on the terminal arm, P. If P is a point on the circumference of a circle of radius r, the ratios would be defined as follows.

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$

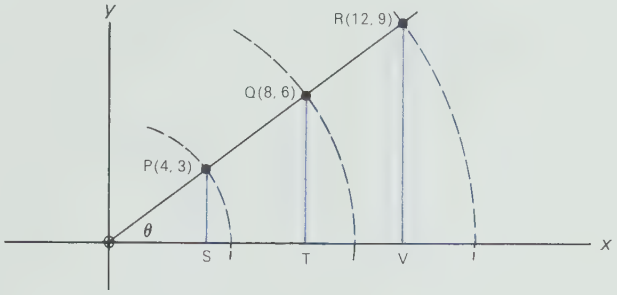


Notice that these values do not depend on the point used on the terminal arm.

$$\sin\theta = \frac{3}{5} = \frac{6}{10} = \frac{9}{15}$$

$$\cos\theta = \frac{4}{5} = \frac{8}{10} = \frac{12}{15}$$

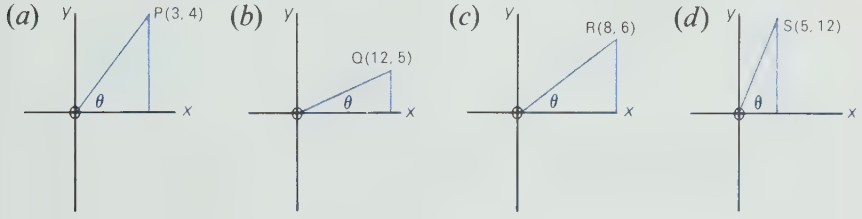
$$\tan\theta = \frac{3}{4} = \frac{6}{8} = \frac{9}{12}$$



By applying these trigonometric values, we can find unknown sides and angles of a triangle, as the following examples show.

Example 1

State the value of x, y, and r and then the sine and cosine of the angle indicated in each diagram.



Solution

	(a)	(b)	(c)	(d)
x	3	12	8	5
y	4	5	6	12
r	5	13	10	13
$\sin\theta$	$\frac{4}{5}$	$\frac{5}{13}$	$\frac{6}{10} = \frac{3}{5}$	$\frac{12}{13}$
$\cos\theta$	$\frac{3}{5}$	$\frac{12}{13}$	$\frac{8}{10} = \frac{4}{5}$	$\frac{5}{13}$
$\tan\theta$	$\frac{4}{3}$	$\frac{5}{12}$	$\frac{6}{8} = \frac{3}{4}$	$\frac{12}{5}$

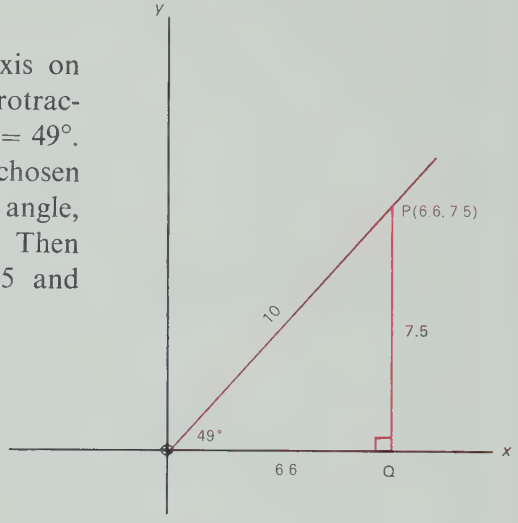
Pythagoras!
$3^2 + 4^2 = 5^2$
$8^2 + 6^2 = 10^2$
$9^2 + 12^2 = 15^2$

Example 2

Draw an angle of 49° and finds its sine, cosine, and tangent by measurement.

Solution

Mark the coordinate axis on graph paper and with a protractor make an angle QOP = 49°. Since any point may be chosen on the terminal arm of the angle, choose OP = 10 units. Then by measurement PQ = 7.5 and OQ = 6.6.



$$\sin 49° \doteq \frac{7.5}{10} = 0.75$$

$$\cos 49° \doteq \frac{6.6}{10} = 0.66$$

$$\tan 49° \doteq \frac{7.5}{6.6} = 1.14$$

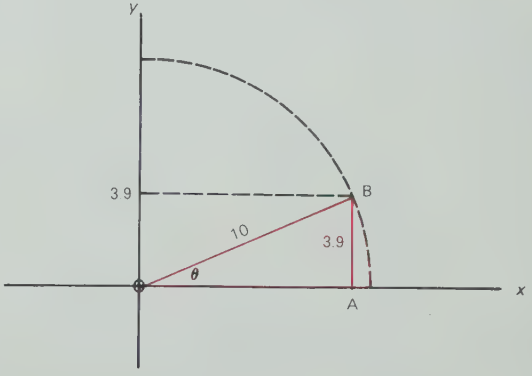
Why choose OP = 10?

$\sin\theta = \frac{opp}{hyp}$

Example 3

Construct an angle whose sine is 0.39 and find the approximate value of its cosine.

Solution



Since $0.39 = \frac{3.9}{10} = \sin\theta$, with centre the origin draw an arc of a circle with radius 10 units. On the y axis measure off 3.9 units and find the point B on the arc.

$\therefore \angle AOB = \theta$, the required angle.

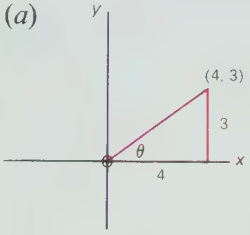
$$\therefore \cos\theta = \frac{OA}{OB} = \frac{9.2}{10} = 0.92$$

Exercise 15.1

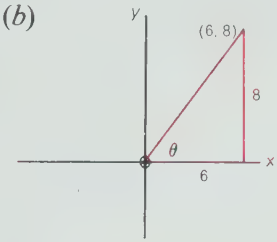
Discussion

1. State the values of sin θ, cos θ, and tan θ in each of the following diagrams.

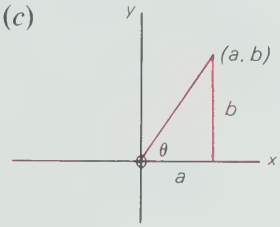
(*a*)



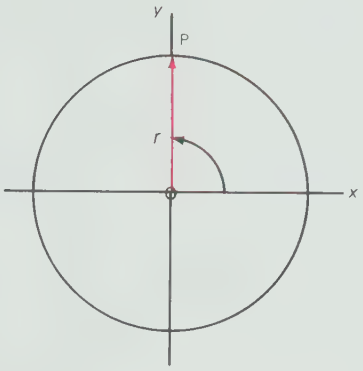
(*b*)



(*c*)



2. (*a*) For a 90° angle, if the radius is r, name the x and y coordinates of P.
   (*b*) Find sin 90° and cos 90°.
   (*c*) As θ goes from 0° to 360°, what is the largest value of x? Of y?
   (*d*) What is the greatest value of sin θ and cos θ?



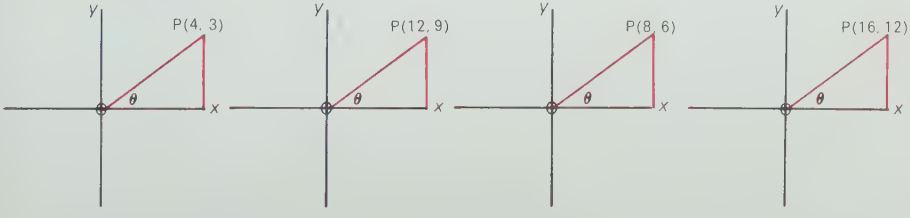
Written Solutions

3. Draw an angle of 77° and find, by measurement, the value of its sine and cosine.

4. The sine of an angle is 0.88. Draw the angle and find the value of its cosine.

5. Construct an angle A such that tan A = 1.6. Measure the angle to the nearest degree and find its sine and cosine.

6. Find the sine, cosine, and tangent of θ in the following diagrams.



7. What do your answers to Question 6 indicate about the trig values of angles of the same size?

$\sin\theta = \frac{\text{opp}}{\text{hyp}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$

3. sin 77° = 0.97,
   cos 77° = 0.22

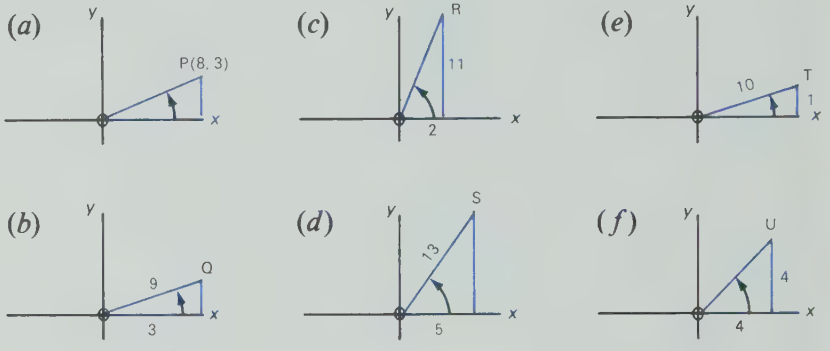
6.(a) sin θ = 0.6
   cos θ = 0.8
   tan θ = 0.75
   (b) sin θ = 0.6
   cos θ = 0.8
   tan θ = 0.75

8. Construct $\triangle ABC$ such that $\angle C$ is a right angle, the hypotenuse is 10 units in length, and $\tan A = 0.81$. Measure AC and the angle A. Find the values of sin A and cos A.

9. Draw angles whose sines are 0.67 and 0.94 in standard position. Measure their difference in degrees.

10. Construct $\triangle PLQ$ such that $\sin P = 0.5$ and $\angle Q$ is a right angle. Measure PQ and the angle P. Find the values of cos P and tan P.

11. Write the sine, cosine, and tangent of each of the angles that are marked in the following diagrams.



8. AC = 7.8, $\angle A = 39°$
sin A = 0.63,
cos A = 0.78

10. PQ = 0.87, $\angle P = 30°$
cos P = 0.87,
tan P = 0.58

11.(b) $\sin = \frac{2\sqrt{2}}{3}$

$\cos = \frac{1}{3}$

$\tan = 2\sqrt{2}$

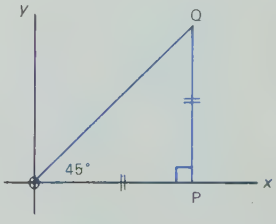
(d) $\sin = \frac{12}{13}$

$\cos = \frac{5}{13}$

$\tan = \frac{12}{5}$

"Trig" is a handy short form of "trigonometric"

15.2 Trig Values of Special Angles



A *Trig Values of 45° Angles*

1. State the measure of $\angle OQP$.

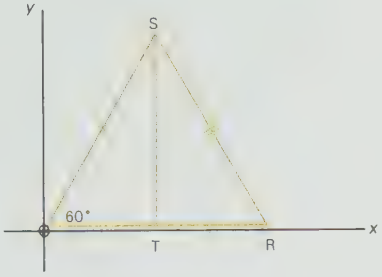
2. OP = PQ. (Why?)
Let OP = PQ = 1 unit
$\therefore$ OQ = $\sqrt{2}$ units (Why?)

3. From the diagram, list the values of sin 45°, cos 45°, and tan 45°.

4. Find the value of each expression.
   (*a*) $\sin 45° \times (\tan 45°)^2$
   (*b*) $\cos \frac{\pi}{4}\text{ rad} \times \sin \frac{\pi}{4}\text{ rad} \times \tan \frac{\pi}{4}\text{ rad}$
   (*c*) $(\sin 45°)^2 + (\cos 45°)^2$

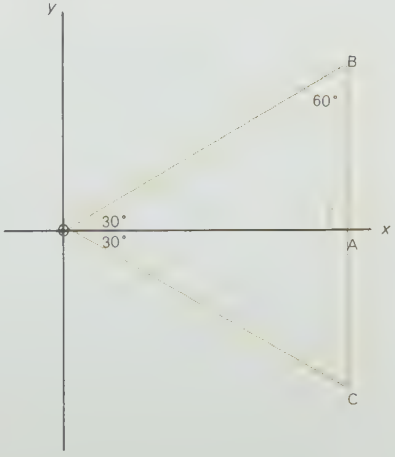
π rad = 180°
$\sqrt{2} \times \sqrt{2} = 2$

B *Trig Values of 60° Angles*



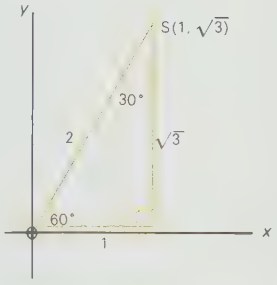
1. Draw an equilateral $\triangle OSR$ on the axes as indicated.
   In $\triangle OSR$, $\angle SOR = \angle ORS = \angle RSO = 60°$
   and $OS = RS$ (Why?)
2. Draw $ST \perp OR$.
   $\therefore OT = TR$ (Why?)
3. Let $OT = 1$ unit. What is the length of TR?
   Then $OS = RS = 2$ units (Why?)
   $OS^2 = OT^2 + ST^2$
   $\therefore \quad 2^2 = 1^2 + ST^2$
   $\therefore ST^2 = 3$
   $\therefore ST = \sqrt{3}$
4. (*a*) Find the coordinates of S.
   (*b*) From the diagram and the coordinates of S, list the three trig values of a 60° angle.
5. Evaluate.
   (*a*) $\sin 60° \times \cos 60° \times \tan 60°$
   (*b*) $(\sin 60°)^2 + (\cos 60°)^2$
   (*c*) $\sin \frac{\pi}{3}\text{ rad} \times \cos \frac{\pi}{3}\text{ rad} \times \left(\tan \frac{\pi}{3}\text{ rad}\right)^2$
   (*d*) $\sin 60° \times \cos 45° \times \cos 60° \times \sin 45°$

C *Trig Values of 30° Angles*

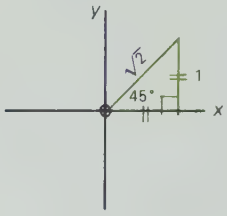


1. Draw a 30° angle in standard position. Complete the diagram as indicated.
   $\triangle OBC$ is equilateral. (Why?)
   Let $OB = 2$ units
   $\therefore BC = 2$ units
   and $BA = 1$ unit
   and $OA = \sqrt{3}$ units as before.
2. Find the coordinates of B.
3. List the three trig values of a 30° angle.

For ⊥ read "is perpendicular to."



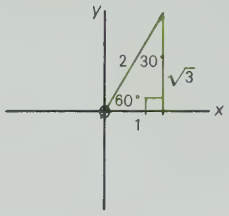
Summary of Trig Values



$$\sin 45° = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{1}{\sqrt{2}}$$

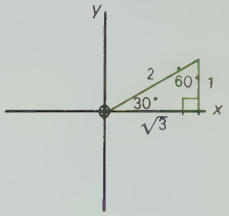
$$\tan 45° = \frac{1}{1}$$



$$\sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{1}{2}$$

$$\tan 60° = \frac{\sqrt{3}}{1}$$



$$\sin 30° = \frac{1}{2}$$

$$\cos 30° = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{1}{\sqrt{3}}$$

Exercise 15.2

Written Solutions

1. Evaluate.
   (*a*) $\sin 30° \times \cos 30° \times \tan 30°$
   (*b*) $(\sin 30°)^2 + (\cos 30°)^2$
   (*c*) $2 \sin \frac{\pi}{6}\text{ rad} + \tan \frac{\pi}{6}\text{ rad} + 3 \cos \frac{\pi}{6}\text{ rad}$
   (*d*) $\sin \frac{\pi}{6}\text{ rad} \times \cos \frac{\pi}{3}\text{ rad} \times \cos \frac{\pi}{6}\text{ rad} \times \sin \frac{\pi}{3}\text{ rad}$

2. Evaluate.
   (*a*) $2\sin 45° + \tan 45°$
   (*b*) $\tan 30° \times \tan 45° \times \tan 60°$
   (*c*) $\sin 30° \times \sin 45° \times \sin 60°$
   (*d*) $\cos 45° \times \cos 60° \times \cos 30°$
   (*e*) $\cos^2 30° + \sin^2 30°$
   (*f*) $\tan^2 \frac{\pi}{6}\text{ rad} + 2 \sin \frac{\pi}{3}\text{ rad} + \tan \frac{\pi}{4}\text{ rad} - \tan \frac{\pi}{6}\text{ rad} + \cos^2 \frac{\pi}{6}\text{ rad}$
   (*g*) $\cos 60° - \tan^2 45° + \frac{3}{4}\tan^2 30° + \cos 30° - \sin 30°$
   (*h*) $\left(\sin \frac{\pi}{6}\text{ rad} + \cos \frac{\pi}{6}\text{ rad}\right)\left(\sin \frac{\pi}{3}\text{ rad} - \cos \frac{\pi}{3}\text{ rad}\right)\left(\frac{1}{\cos \frac{\pi}{3}\text{ rad}}\right)$
   (*i*) $\tan^2 45° \times \sin 60° \times \tan 30° \times \tan^2 60°$
   (*j*) $\cos^2 30° - \cos^2 45°$

Note:
$\sin^2\theta = (\sin\theta)^2$

1.(a) $\frac{1}{4}$
(b) 1
2.(a) $\sqrt{2} + 1$
(b) 1
(f) $\frac{25 + 8\sqrt{3}}{12}$
(g) $\frac{2\sqrt{3} - 3}{4}$

3. Show that each statement is true.
   (*a*) $\sin 60° = \cos 60° \times \tan 60°$
   (*b*) $\sin 60° = 2 \sin 30° \times \cos 30°$
   (*c*) $1 = 2 \sin 45° \times \cos 45°$
   (*d*) $\sin 60° \times \cos 30° + \cos 60° \times \sin 30° = 1$
   (*e*) $\tan 60° = \dfrac{2 \tan 30°}{1 - \tan^2 30°}$

15.3 The Use of Trig Tables

A In the previous section we established the sine, cosine, and tangent values of 30°, 45°, and 60° angles. Tables have been constructed that allow us to find the trig values of all angles between 0° and 90°. These values have all been rounded off to 4 decimal places, thus, most of our trig values are approximations. You will find that these tables agree with the trig values we found before. For example,

$$\sin 30° = \frac{1}{2} = 0.5$$

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.4142}{2} = 0.7071.$$

1. From the tables find the values of sin 37° and sin 77°.
   (*a*) Opposite 37° and under the column headed sin θ, we find sin 37° = 0.6018.
   (*b*) Opposite 77° and under the column headed sin θ, we find sin 77° = 0.9744.
2. From the tables find the values of sin 60° and sin 11°.
3. Use the tables to find the values of cos 63° and tan 87°.
   (*a*) Opposite 63° and under the column headed cos θ, we find cos 63° = 0.4540.
   (*b*) Opposite 87° and under the column headed tan θ, we find tan 87° = 19.081.
4. From the tables find the values of cos 12°, tan 12°, cos 83°, tan 45°.

Pretty good "approximations" I'd say!

$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

See pages 462, 463

θ	sin θ
0	.0000
1	.0175
⋮	⋮
36	.5878
37	.6018
38	.6157
⋮	⋮
75	.9659
76	.9703
77	.9744

θ	cos θ
⋮	⋮
20	.9397
21	.9336
22	.9272
23	.9205

See pages 462, 463

B Sometimes the problem is just the reverse. We know the value of a trig ratio, but we do not know the value of the angle.

1. From the tables find θ if $\cos \theta = 0.9336$. Look under the column headed $\cos \theta$ and opposite the row headed 21°.
$$\therefore \theta = 21°$$
2. If $\tan A = 6.3138$ find A. 6.3138 is found under the column headed $\tan \theta$. It is opposite the row headed 81°.
$$\therefore A = 81°$$
3. Find the angle A.
   (*a*) $\sin A = 0.1564$
   (*b*) $\cos A = 0.6691$
   (*c*) $\tan A = 0.0699$

Exercise 15.3

Discussion

1. By looking in the tables find the angle θ for which both sine and cosine values are equal.
2. For what angles are the tangent values greater than 1? Less than 1?
3. What happens to the sine values as θ increases from 0° to 90°?
4. What happens to the cosine values as θ increases from 0° to 90°?

Written Solutions

5. Use tables to evaluate.

(*a*) sin 67°	(*g*) cos 60°	(*m*) cos 26°
(*b*) cos 22°	(*h*) sin 30°	(*n*) cos 40°
(*c*) tan 22°	(*i*) tan 45°	(*o*) cos 85°
(*d*) sin 89°	(*j*) sin 17°	(*p*) sin 8°
(*e*) cos 2°	(*k*) sin 26°	(*q*) tan 84°
(*f*) sin 35°	(*l*) cos 17°	(*r*) tan 4°

6. From the tables find the following angles.

(*a*) $\sin \theta = 0.8829$	(*e*) $\tan A = 0.5774$
(*b*) $\cos A = 0.8090$	(*f*) $\cos \theta = 0.4067$
(*c*) $\tan A = 5.1446$	(*g*) $\tan C = 0.0699$
(*d*) $\sin B = 0.4067$	(*h*) $\sin B = 0.0698$

SOH
CAH
TOA

5.(a) 0.9205 (j) 0.2924
(b) 0.9272 (k) 0.4384
(c) 0.4040 (l) 0.9563
(h) 0.5000 (q) 9.5144

6.(b) 36° (f) 66°
(c) 79° (g) 4°

15.4 Solution of Right Triangles

A Trigonometric values are used to find unknown sides and angles of right triangles. Therefore many problems in surveying and navigation can be solved by the application of trigonometry to heights and distances that cannot be measured directly.

A triangle is "solved" when you can state the lengths of the three sides and the measures of the three angles.

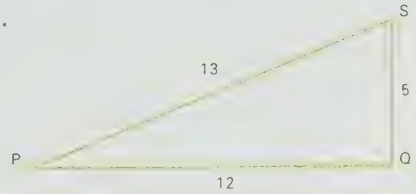
$\sin\theta = \frac{opp}{hyp}$

$\cos\theta = \frac{adj}{hyp}$

$\tan\theta = \frac{opp}{adj}$

Example 1

Find $\angle P$ in the given triangle.



Solution

$$\cos P = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13}$$

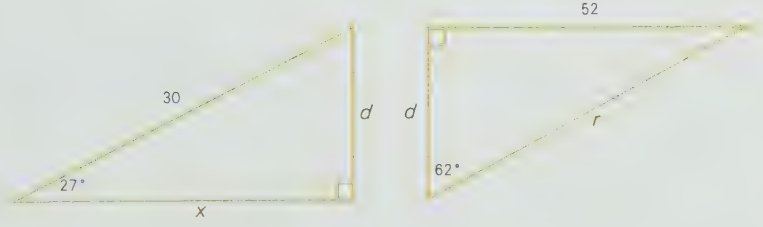
$\therefore \cos P = 0.9231$

$\therefore \quad \angle P = 23°$

Example 2

Find the length d in each of the following triangles.

(*a*) (*b*)



Solution

(*a*) $\sin 27° = \frac{\text{opp}}{\text{hyp}} = \frac{d}{30}$

$\therefore d = 30 \sin 27°$

$= 30(0.4540)$

$= 13.6200$

$\therefore d = 13.62$

(*b*) $\tan 62° = \frac{\text{opp}}{\text{adj}} = \frac{52}{d}$

$\therefore d \tan 62° = 52$

$\therefore d = \frac{52}{\tan 62°}$

$= \frac{52}{1.8807}$

$\therefore d = 27.6$

1. Find $\angle P$ in each of the following diagrams by using a trig value.

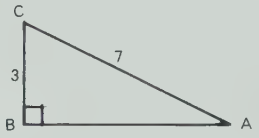
(*a*)

(*b*)

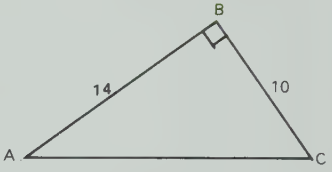
(*c*)

2. Find $\angle A$ in each of the following.

(*a*)

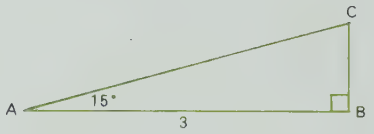


(*b*)

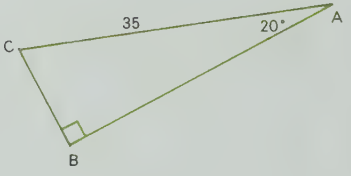


3. Find BC in each of the following.

(*a*)



(*b*)

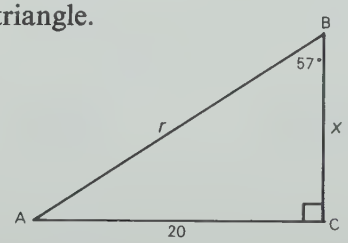


To solve a triangle means to find all sides and angles.

B Once we can find the lengths of unknown sides and the measures of unknown angles, we are in a position to solve triangles.

Example 1

In $\triangle ABC$, $\angle C = 90°$, $\angle B = 57°$, and $AC = 20$ units. Solve the triangle.



We have to find $\angle A$, r, and x.

Solution

$$\begin{aligned}\angle A &= 180° - (90° + 57°)\\ &= 180° - 147°\\ &= 33°\end{aligned}$$

$$\sin B = \frac{\text{opp}}{\text{hyp}} \qquad \cos B = \frac{\text{adj}}{\text{hyp}}$$

$$\therefore \sin 57° = \frac{20}{\text{hyp}} \qquad \therefore \cos 57° = \frac{\text{adj}}{\text{hyp}}$$

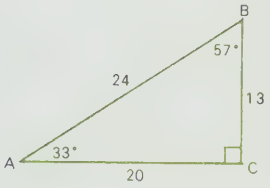
$$\therefore r \sin 57° = 20 \qquad \therefore r \cos 57° = x$$

$$\therefore r\,(0.8387) = 20 \qquad \therefore x = 24\,(0.5446)$$

$$r = \frac{20}{0.8387} \qquad \doteq 13 \text{ units}$$

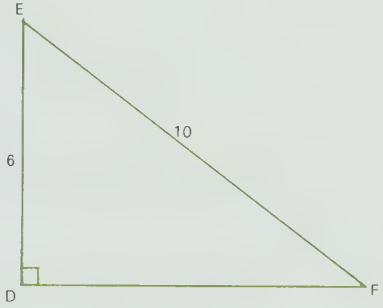
$$\doteq 24 \text{ units}$$

$\therefore$ the triangle is solved.



Example 2

Given $\triangle DEF$ with $\angle D = 90°$, DE = 6 units, and EF = 10 units. Solve the triangle.



Solution

$$\sin \angle F = \frac{6}{10} = 0.6$$

$$\therefore \angle F \doteq 37°$$

$$\therefore \angle E = 180° - (90 + 37)°$$

$$= 53°$$

$$\text{Also } DF^2 = 10^2 - 6^2$$

$$= 100 - 36$$

$$= 64$$

$$\therefore DF = 8 \text{ units}$$

$\therefore$ the triangle is solved.

$\sin \theta = \frac{opp}{hyp}$

Exercise 15.4

Written Solutions

Solve the triangles when the following angles and sides are given.

1. $\angle A = 90°$, AB = 37, BC = 100
2. $\angle A = 90°$, BC = 17, AC = 11
3. $\angle B = 90°$, $\angle C = 63°$, BC = 20
4. $\angle B = 90°$, $\angle C = 37°$, AC = 100
5. $\angle C = 90°$, BC = 225, AC = 272
6. $\angle A = 90°$, $\angle B = 36°$, AB = 100

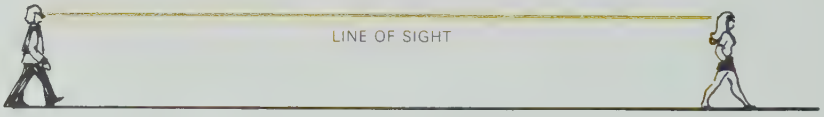
1. $\angle B = 68°$, $\angle C = 22°$, AC = 93
2. $\angle C = 50°$, $\angle B = 40°$, AB = 13

15.5 Problems Involving Right-Angled Triangles

It is essential in solving practical problems involving right-angled triangles that you make a clearly-labelled diagram. The problem then becomes similar to those of the previous section where you solved the right-angled triangles for the unknown angles and sides.

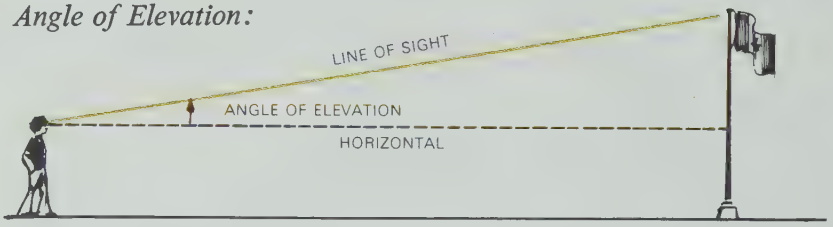
Before looking at some examples, some basic concepts must be clearly established so that the wording in the problems can be understood.

Line of Sight: The straight line joining the observer's eye to the object under observation.



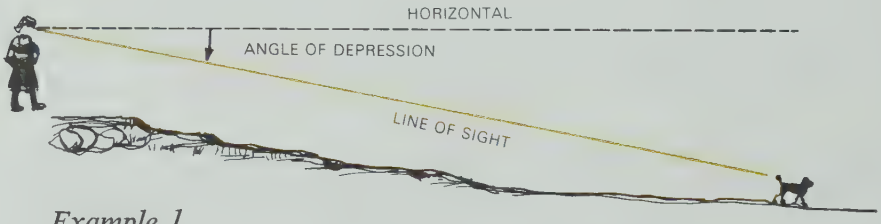
Angle of Elevation:

You raise your eyes.



Angle of Depression:

You lower your eyes.

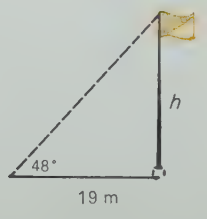


Example 1

The angle of elevation of the top of a flagpole is 48°. Determine the height of the flagpole if the angle of elevation was measured 19 m from the centre of its base.

$\tan\theta = \frac{\text{opp}}{\text{adj}}$

Solution



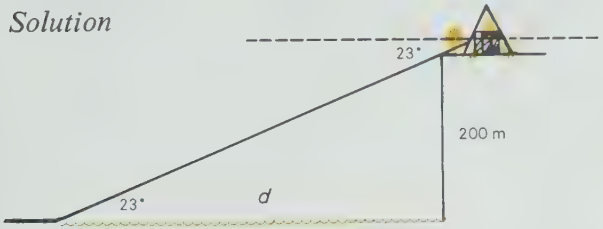
$$\tan 48° = \frac{h}{19}$$
$$\therefore 19(\tan 48°) = h$$
$$\therefore h \doteq 19(1.1106)$$
$$\doteq 21$$

$\therefore$ the height of the flagpole is approximately 21 m.

Example 2

The angle of depression of a raft on a lake is 23° from a cottage on a cliff that is 200 m above the water level of the lake. Find the distance of the raft from the base of the cliff.

Solution



$$\frac{200}{d} = \tan 23°$$

$$d \tan 23° = 200$$

$$\therefore d \doteq \frac{200}{0.4245}$$

$$\doteq 470$$

$\therefore$ the raft is approximately 470 m from the base of the cliff.

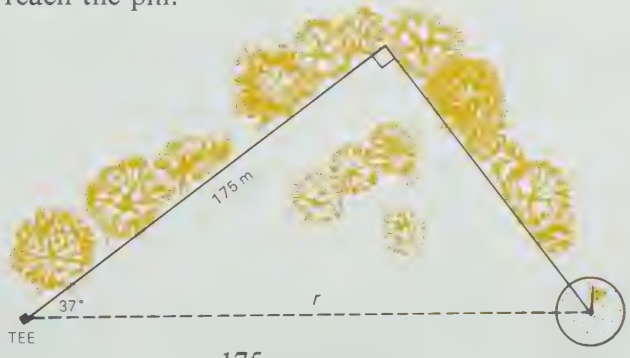
$\tan \theta = \frac{opp}{adj}$

$\cot \theta = \frac{adj}{opp}$

Example 3

On a "dogleg," Dan Duffer decides to drive straight for the flag, as shown. The scorecard tells him that it is 175 m down the middle of the fairway to the corner of the dogleg. Dan estimates that the angle from the tee to the pin is 37°. Find the length of his drive in order to reach the pin.

Solution

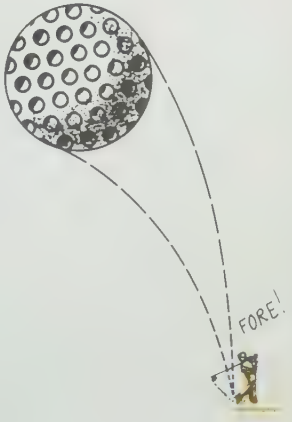


$$\frac{175}{r} = \cos 37°$$

$$\therefore r \doteq \frac{175}{0.7986}$$

$$\doteq 220$$

$\therefore$ Dan must drive 220 m to reach the pin on the green.

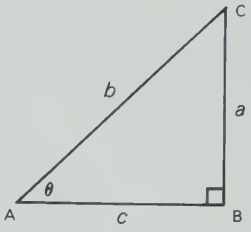


Exercise 15.5

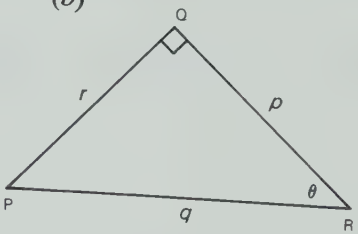
Discussion

1. For each marked angle,
   —name the *opposite* and *adjacent* sides
   —state the sine, cosine, and tangent of the angle.

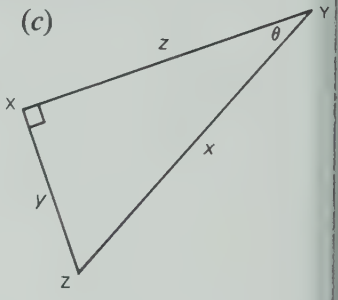
(*a*)



(*b*)

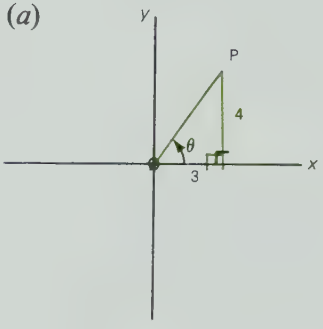


(*c*)

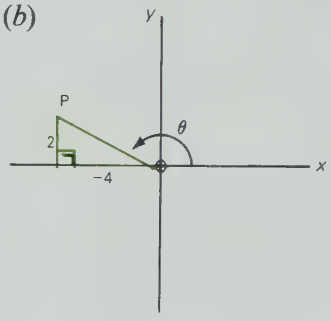


2. State the coordinates of P in each diagram.

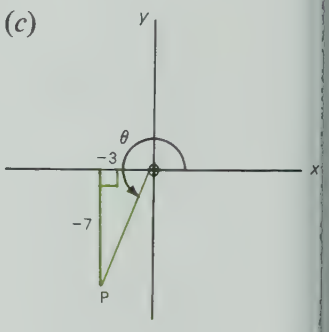
(*a*)



(*b*)



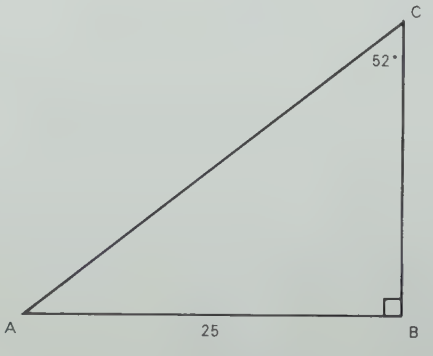
(*c*)



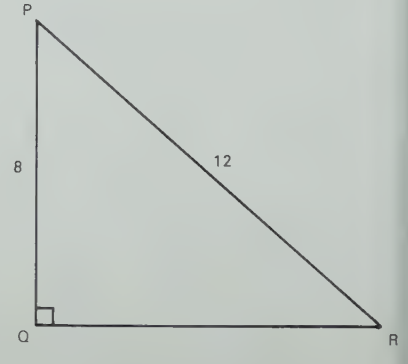
3. State the sine, cosine, and tangent of each θ in Question 2.

4. Explain how you would solve these triangles.

(*a*)



(*b*)



Written Solutions

5. The angle of elevation of the top of a tower at a distance of 300 m from its base is 30°. Find the height of the tower.

6. A kite held by a rope 200 m long makes an angle of elevation with the ground of 54°. How high is the kite above the ground?

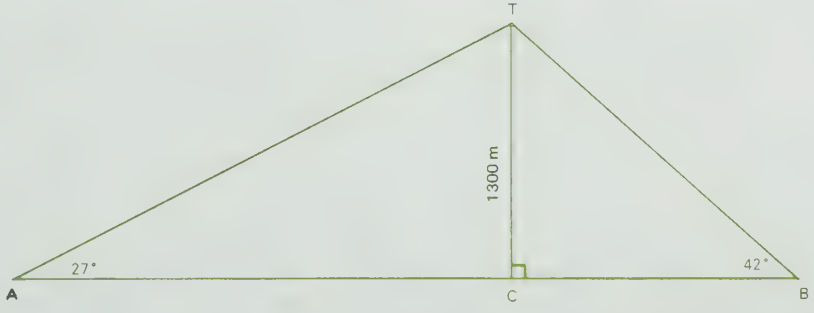
7. A construction engineer is told that a road will rise 27 m over a 160 m distance measured along the road. Calculate the angle of elevation of the road.

8. What angle will a 25 m guy wire make with a TV tower when the wire is anchored at a point 15 m from the base of the tower?

9. A ski slope at a local resort is described as having a vertical drop of 165 m during a run of 600 m. Find the angle of elevation of the ski slope.

10. The angle of elevation of a highway is 6°. Find the rise for 150 m measured along the roadway.

11. A tunnel is to be made through a hill. The angles of elevation to the top of the hill from points on opposite sides of the hill but on the same level are 27° and 42°. The top of the hill is 1300 m above this. Find the length of the tunnel joining these two points.

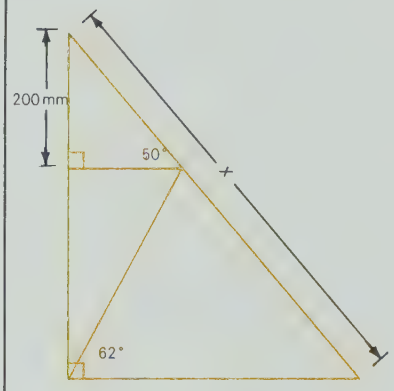


12. A street is 12 m wide. On one side is a pole 13 m high and on the other a pole 21 m high. Find the angle that the wire joining the tops of the poles will make with a horizontal wire from the 13 m pole.

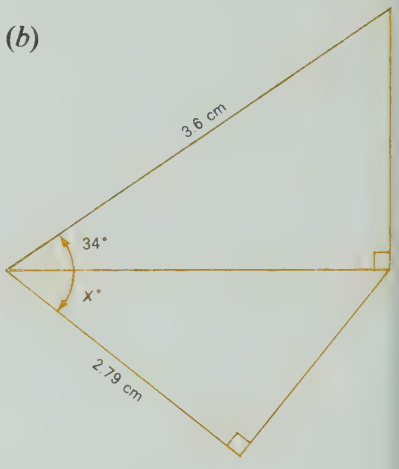
5. 173 m
7. 10°
9. 16°
11. 3995 m

13. The following problems are simplifications of various machine and tool design problems. Find the value of x to three significant figures or to the nearest degree.

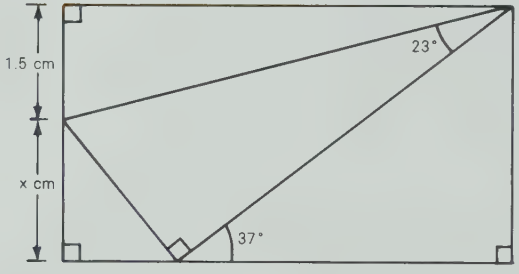
(*a*)



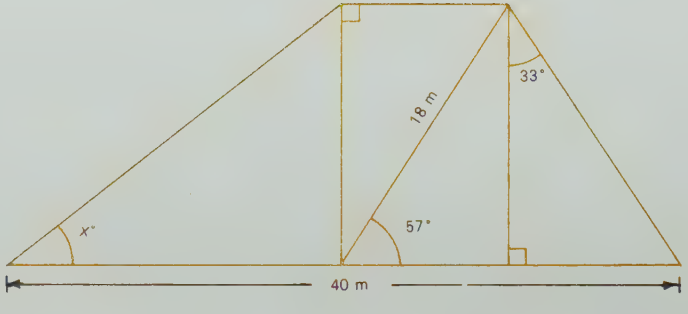
(*b*)



(*c*)



(*d*)



13.(a) 673 mm
(b) 21°

Continued Fractions

Sometimes a number can be represented by a *continued fraction.*

$$\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+1}}}$$

Continued fractions may be simplified like this.

1. Start at the lowest line $3 + 1 = 4$

$$\therefore \text{ the fraction is } \cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{4}}}$$

2. Again consider the lowest line. $3 + \frac{1}{4}$ could be written as $\frac{12}{4} + \frac{1}{4} = \frac{13}{4}$, $\therefore$ the fraction is $\cfrac{1}{3+\cfrac{1}{\frac{13}{4}}}$

3. Simplify $\cfrac{1}{\frac{13}{4}}$. $1 \div \frac{13}{4} = 1 \times \frac{4}{13} = \frac{4}{13}$

$$\therefore \text{ the fraction is } \cfrac{1}{3+\cfrac{4}{13}}$$

4. By simplifying the denominator $3 + \frac{4}{13}$ we get $\frac{39}{13} + \frac{4}{13} = \frac{43}{13}$

$$\therefore \text{ the fraction is } \cfrac{1}{\frac{43}{13}}$$

5. Again by simplifying $1 \div \frac{43}{13}$ we get the answer $\frac{13}{43}$.

Now try these.

Reduce the following continued fractions to their simplest form.

1. $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}}$

2. $\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}}$

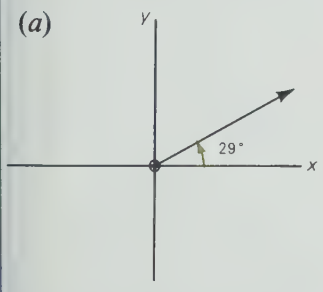
Unit 16
Trigonometric Values of Any Angle

16.1 Get-Set

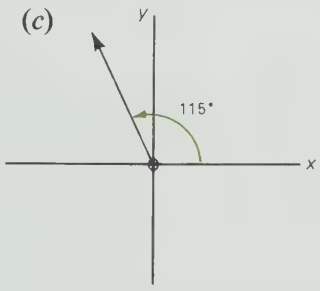
A Do you recall the sign conventions for angles?

1. Determine if the value of each angle is positive or negative.

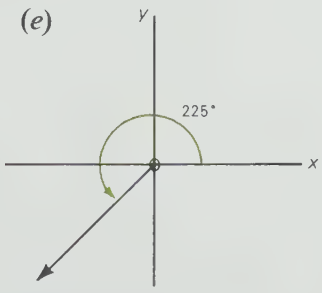
(*a*)



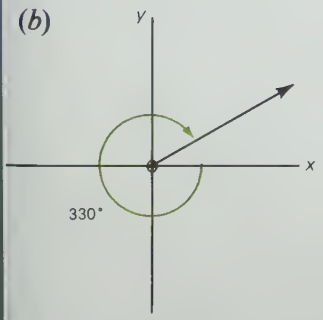
(*c*)



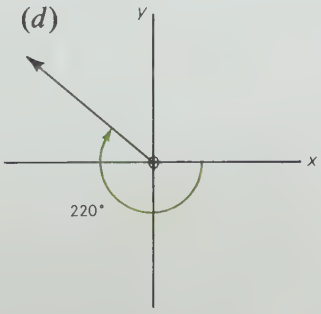
(*e*)



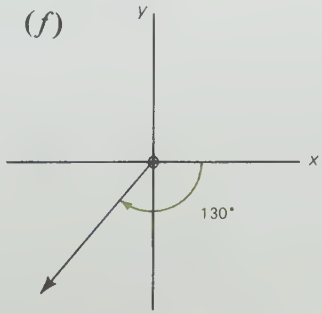
(*b*)

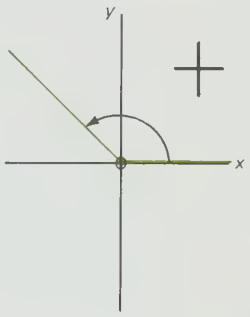


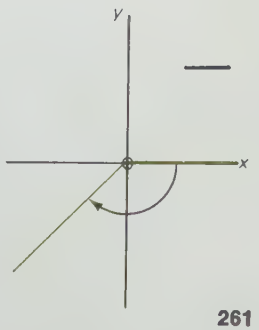
(*d*)



(*f*)

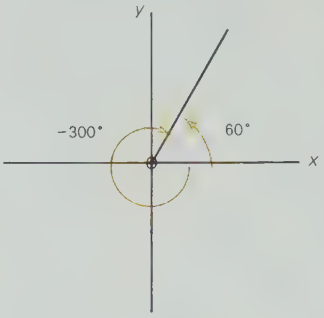




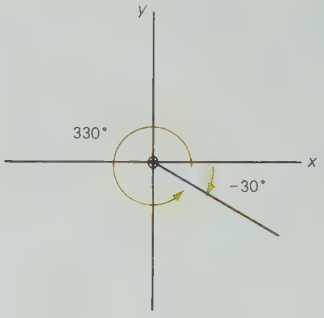


One good turn deserves another.

B Also recall that angles having the same terminal arm are called *coterminal angles.*



$-300°$ and $60°$ are coterminal angles.



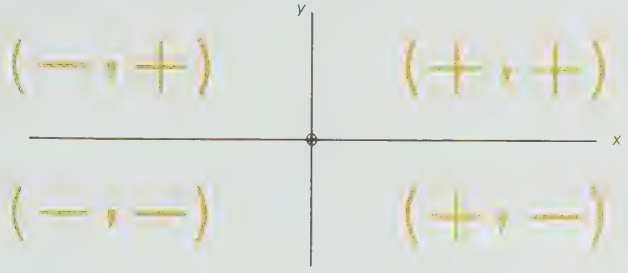
$330°$ and $-30°$ are coterminal angles.

1. Which of the following angles are coterminal with an angle of $10°$ if all are in standard position?
   (*a*) $370°$
   (*b*) $190°$
   (*c*) $-350°$
   (*d*) $730°$
   (*e*) $170°$
   (*f*) $-710°$
   (*g*) $-10°$
   (*h*) $350°$

C The Cartesian plane is divided into four quadrants by the x and y axes.

1. Sketch angles with the following measurements and state the quadrant in which the terminal arm lies.
   (*a*) $-155°$
   (*b*) $32°$
   (*c*) $-260°$
   (*d*) $-320°$
   (*e*) $-62°$
   (*f*) $135°$
   (*g*) $240°$
   (*h*) $340°$

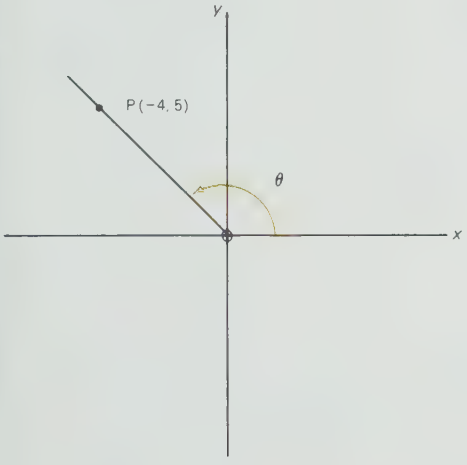
D The coordinates of a point on the terminal arm of an angle in standard position also determine the quadrant in which the angle lies. Recall the signs of the coordinates.



Example

A point P(−4, 5) lies on the terminal arm of $\angle\theta$ in standard position. Determine the quadrant in which the angle lies.

Solution



Since P lies on the terminal arm and is in the 2nd quadrant, $\angle\theta$ is a 2nd-quadrant angle.

Recall that "standard position" means that the initial arm of the angle lies along the positive x axis.

1. Determine the quadrant in which the terminal arm lies if each of the following points lies on the terminal arm.

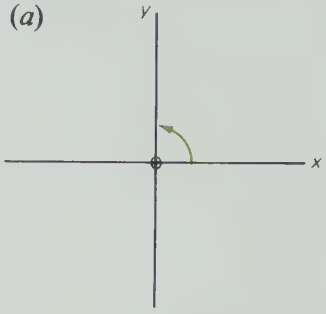
(*a*) (−3, −4)	(*e*) (−8, 10)
(*b*) (2, −5)	(*f*) (4, −5)
(*c*) (−2, 5)	(*g*) (−6, −5)
(*d*) (3, 7)	(*h*) (7, 2)

Exercise 16.1

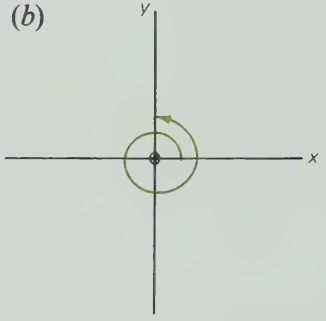
Discussion

1. State the measure in degrees of each angle indicated.

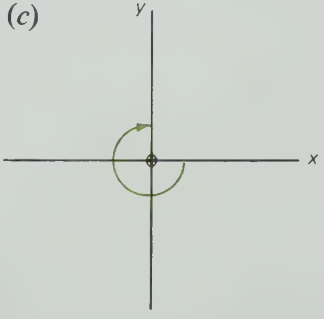
(a)



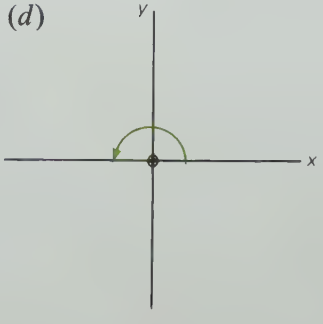
(b)



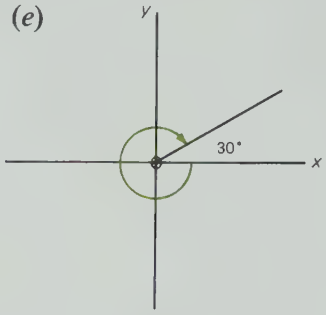
(c)



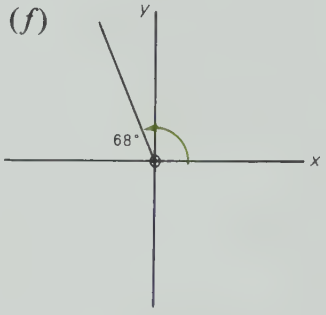
(d)



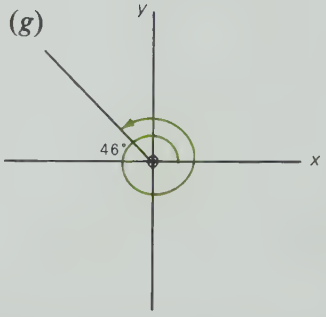
(e)



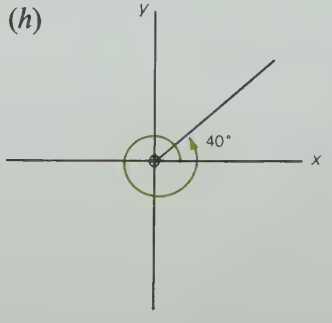
(f)



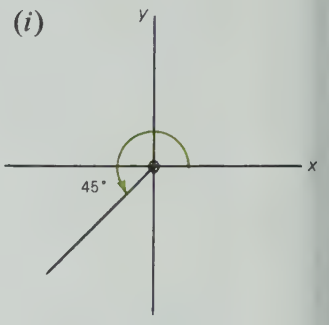
(g)



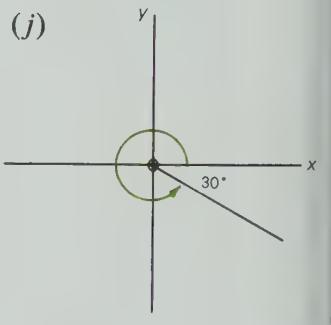
(h)



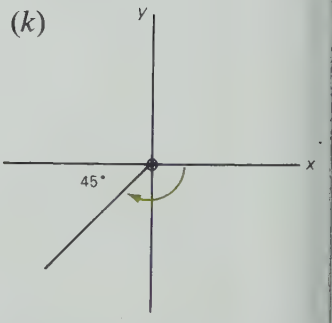
(i)



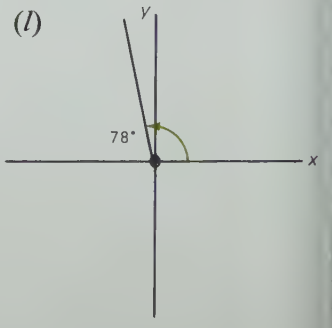
(j)



(k)



(l)



I'm getting dizzy!

Written Solutions

2. Draw each of the following angles in standard position and show the direction of rotation.

(*a*) 45°	(*e*) 135°	(*i*) −210°
(*b*) 300°	(*f*) −90°	(*j*) −315°
(*c*) 30°	(*g*) −135°	(*k*) −360°
(*d*) −60°	(*h*) 150°	(*l*) 225°

3. Which of the following angles are coterminal with an angle of 30° if all are in standard position?

(*a*) 390°	(*e*) 190°
(*b*) 210°	(*f*) −690°
(*c*) −330°	(*g*) −10°
(*d*) 750°	(*h*) −350°

4. Determine the measure of one positive and one negative angle that are coterminal with each of the following angles.

(*a*) −60°	(*e*) 7°
(*b*) −120°	(*f*) 730°
(*c*) 137°	(*g*) −430°
(*d*) −321°	(*h*) −600°

5. Each of the following angles is in standard position. In which quadrant do the terminal arms lie?

(*a*) 80°	(*e*) 223°
(*b*) −120°	(*f*) −8°
(*c*) 333°	(*g*) 188°
(*d*) −253°	(*h*) −300°

6. Each of the following points lies on the terminal arm of an angle in standard position. Determine the quadrant in which the terminal arm lies.

(*a*) (−3, 2)	(*e*) (4, 2)
(*b*) (−7, −8)	(*f*) (6, −1)
(*c*) (7, −5)	(*g*) (−4, 5)
(*d*) (−3, −5)	(*h*) (5, −2)

Recall: "Coterminal" means having the same initial and terminal arms.

4.(a) 300°, −420°
(b) 240°, −480°
(e) 367°, −353°
(f) 10°, −350°

5.(b) 3rd
(c) 4th
(f) 4th
(g) 3rd

6.(c) 4th
(d) 3rd
(g) 2nd
(h) 4th

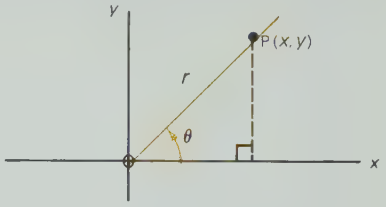
16.2 Trig Values of Any Angle

A We defined the trig value of an angle in terms of the coordinates of a point on the terminal arm.

$$\sin\theta = \frac{y}{r}$$

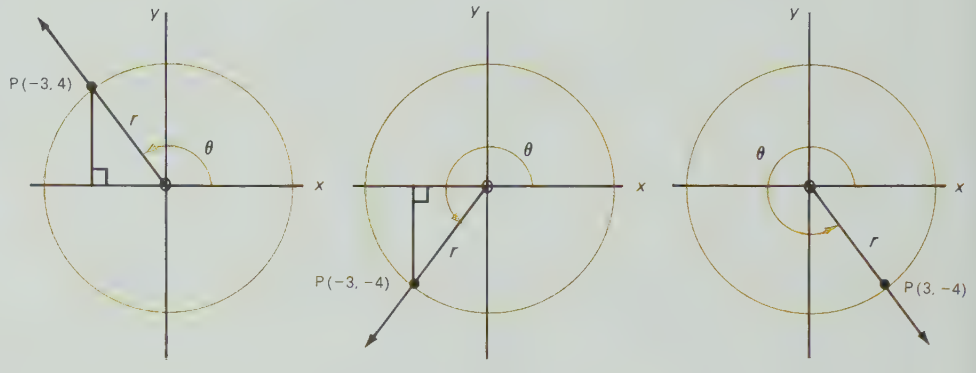
$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$



We shall use this definition for *any* angle θ. We consider r to be always positive. However, x and y may be either positive or negative, depending upon the quadrant.

SOH
CAH
TOA



In each case $x^2 + y^2 = r^2$

$\therefore 9 + 16 = r^2$

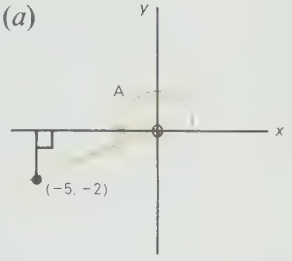
$\therefore 25 = r^2$

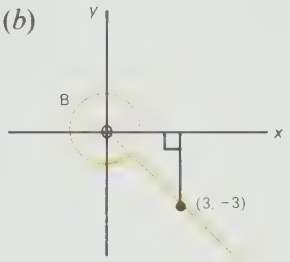
$\therefore r = 5$ (always positive)

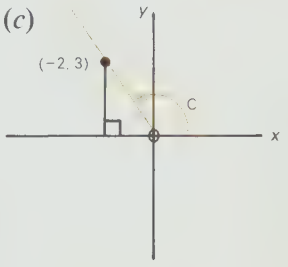
In (*a*)	In (*b*)	In (*c*)
$\sin\theta = \frac{4}{5}$	$\sin\theta = -\frac{4}{5}$	$\sin\theta = -\frac{4}{5}$
$\cos\theta = -\frac{3}{5}$	$\cos\theta = -\frac{3}{5}$	$\cos\theta = \frac{3}{5}$
$\tan\theta = \frac{4}{-3}$	$\tan\theta = \frac{-4}{-3}$	$\tan\theta = -\frac{4}{3}$
$= -\frac{4}{3}$	$= \frac{4}{3}$	

You can see how the signs of the coordinates of the point on the terminal arm affect the sign of the trig values.

1. Determine the three trig values of each of the margin diagrams.







B If we know one trig value of an angle and the quadrant in which the terminal arm lies, we can determine the other trig values.

Example 1

If A is a third-quadrant angle and $\tan A = \frac{5}{4}$, find the values of sin A and cos A.

Solution

Let P(x,y) be a point on the terminal arm. Since $\angle$A is a third-quadrant angle, both x and y are negative.

Let $x = -4$ and $y = -5$

$\therefore r^2 = (-4)^2 + (-5)^2$

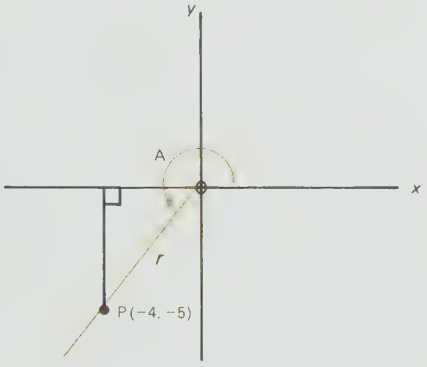
$= 16 + 25$

$= 41$

$\therefore r = \sqrt{41}$

$\therefore \sin A = \frac{-5}{\sqrt{41}}$

and $\cos A = \frac{-4}{\sqrt{41}}$



Example 2

If B is a fourth-quadrant angle and $\cos B = \frac{3}{5}$, find the value of sin B and tan B.

Solution

If $\cos B = \frac{3}{5}$

let $x = 3$

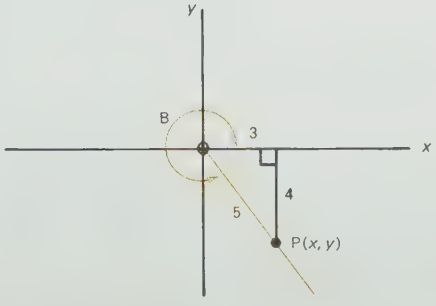
and $r = 5$

$\therefore (3)^2 + y^2 = (5)^2$

$\therefore 9 + y^2 = 25$

$y^2 = 16$

$\therefore y = \pm 4$



Since $\angle$B is in the fourth quadrant, y must equal -4.

$\therefore \sin B = -\frac{4}{5}$ and $\tan B = -\frac{4}{3}$

$\frac{-4}{5} = \frac{4}{-5} = -\frac{4}{5}$

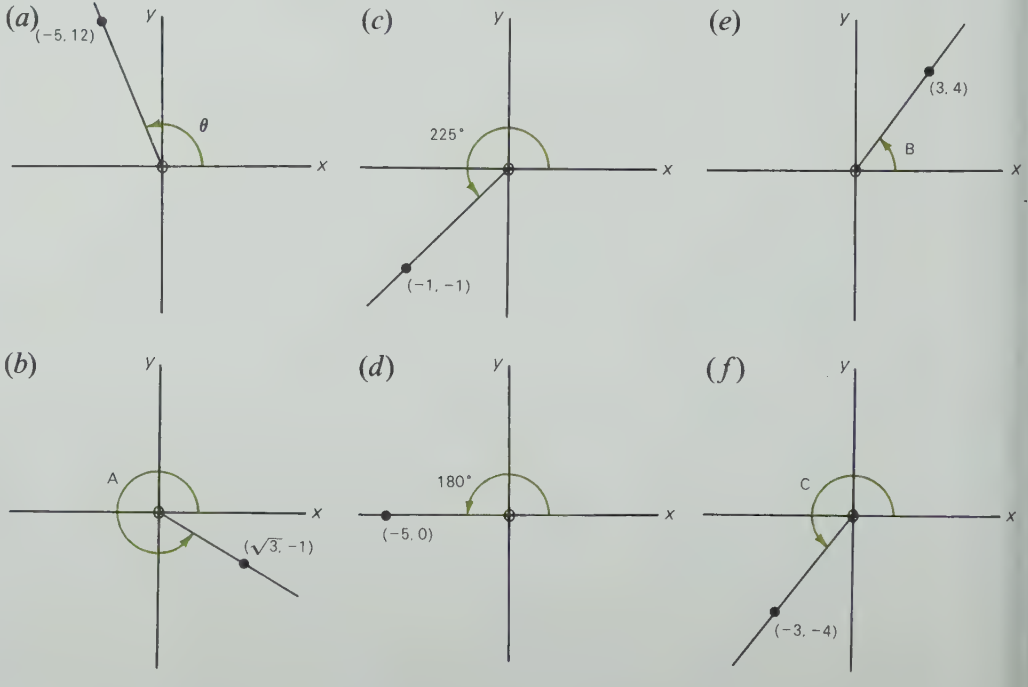
1. Copy the following table and make the necessary calculations to complete it.

Quadrant in which the terminal arm lies	sin A	cos A	tan A
1st	$\frac{6}{10}$		
4th		$\frac{8}{10}$	
3rd			$\frac{4}{3}$
2nd		$\frac{-2}{5}$	

Exercise 16.2

Discussion

1. For each of the following diagrams state the sine, cosine, and tangent for the indicated angle.



Written Solutions

2. Make a diagram whose terminal arm passes through each of the following points.

   (*a*) $(3, 4)$ (*f*) $(5, -12)$
   (*b*) $(-1, 1)$ (*g*) $(-\sqrt{3}, 1)$
   (*c*) $(-6, -8)$ (*h*) $(-2, -7)$
   (*d*) $(-3, -3)$ (*i*) $(-12, -5)$
   (*e*) $(1, -\sqrt{3})$ (*j*) $(8, 15)$

   State the three trig values of each angle.

3. If θ is a third-quadrant angle and $\tan \theta = 1$, state the coordinates of a point on the terminal arm and find $\cos \theta$ and $\sin \theta$.

4. If θ is a second-quadrant angle and $\sin \theta = \frac{5}{13}$, find the values of the following.

   (*a*) $\cos \theta$ (*d*) $\sin^2\theta + \cos^2\theta$
   (*b*) $\tan \theta$ (*e*) $\dfrac{\sin \theta}{\cos \theta}$
   (*c*) $\sin \theta \times \cos \theta$

5. If $\cos \theta = -\frac{3}{5}$, determine possible values of $\sin \theta$ and $\tan \theta$ and sketch the possible angles.

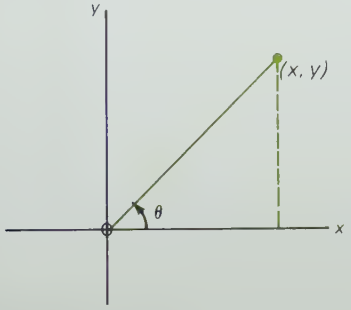
6. If $\tan A = \frac{5}{12}$, sketch the possible angles A and state the values of sin A and cos A.

7. If θ is a third-quadrant angle and $\cos \theta = -\frac{3}{10}$, state the value of $\sin \theta$ and $\tan \theta$.

8. If the terminal arm of angle A passes through $(-5, 12)$, state the value of sin A, cos A, and tan A.

16.3 Signs of Trig Values

1. Copy and complete the table.



1st Quadrant Value	Sign	
	+	−
x		
y		
$\sin \theta$		
$\cos \theta$		
$\tan \theta$		

$\sin \theta = \frac{y}{r}$

$\cos \theta = \frac{x}{r}$

$\tan \theta = \frac{y}{x}$

2.(a) $\sin = \frac{4}{5}$
$\cos = \frac{3}{5}$
$\tan = \frac{4}{3}$

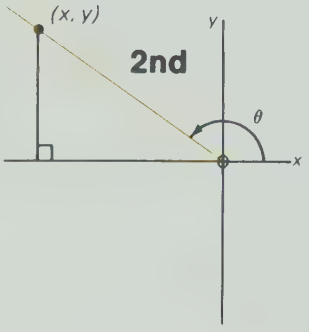
(c) $\sin = -\frac{4}{5}$
$\cos = -\frac{3}{5}$
$\tan = \frac{4}{3}$

4.(a) $-\frac{12}{13}$

(b) $-\frac{5}{12}$

5. $\sin \theta_1 = \frac{4}{5}$, $\tan \theta_1 = -\frac{4}{3}$
$\sin \theta_2 = -\frac{4}{5}$, $\tan \theta_2 = \frac{4}{3}$

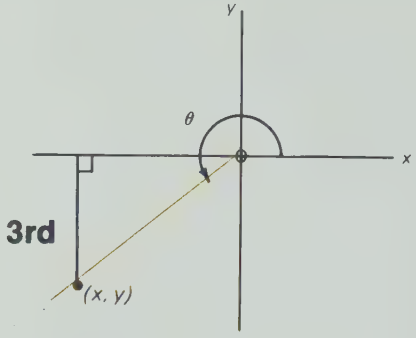
7. $\sin \theta = -\frac{\sqrt{91}}{10}$,
$\tan \theta = \frac{\sqrt{91}}{3}$

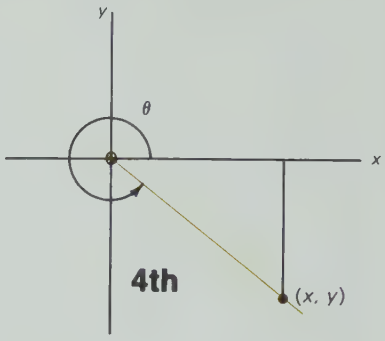


The CAST rule

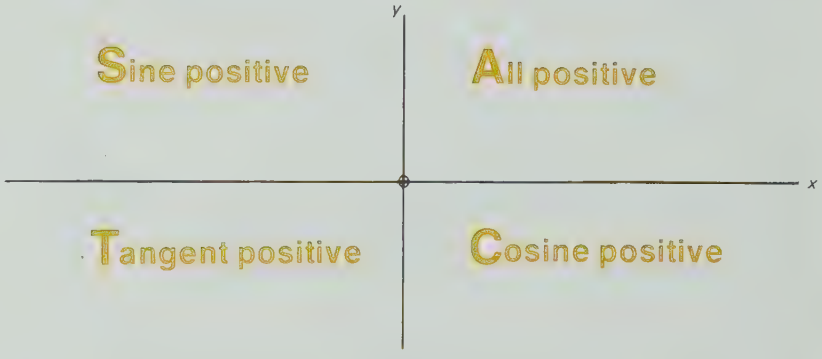
S	A
T	C

2. Using the diagrams, make tables similar to Question 1 for the 2nd, 3rd, and 4th quadrants.





Summary



Exercise 16.3

Written Solutions

1. In what quadrant is angle A, if sin A and cos A are both negative?
2. In what quadrant is angle θ, if sin θ and tan θ are both negative?
3. If angle B is a fourth-quadrant angle, what are the signs of cos B and tan B?
4. In what quadrant is angle C, if sin C is positive and cos C is negative?

5. In each of the following questions a trig value of some angle is given. With the help of a diagram, find all the values for $\sin\theta$, $\cos\theta$, and $\tan\theta$.

(*a*) $\sin\theta = -\frac{4}{5}$

(*b*) $\cos\theta = \frac{4}{5}$

(*c*) $\tan\theta = \frac{1}{3}$

(*d*) $\cos\theta = -\frac{1}{\sqrt{2}}$

(*e*) $\sin\theta = \frac{\sqrt{3}}{2}$

(*f*) $\tan\theta = -1$

(*g*) $\cos\theta = -\frac{\sqrt{3}}{2}$

(*h*) $\tan\theta = -\sqrt{3}$

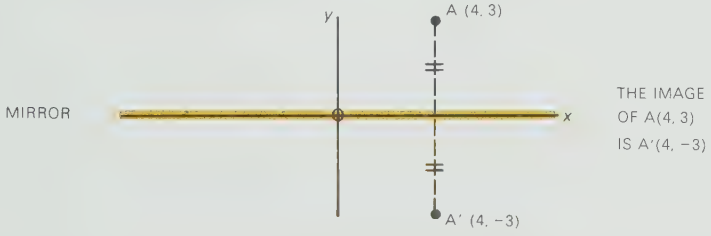
(*i*) $\sin\theta = -\frac{3}{5}$

(*j*) $\cos\theta = -\frac{3}{5}$

5.(b) $\sin = \pm\frac{3}{5}$
$\cos = \frac{4}{5}$
$\tan = \pm\frac{3}{4}$
(e) $\sin = \frac{\sqrt{3}}{2}$
$\cos = \pm\frac{1}{2}$
$\tan = \pm\sqrt{3}$
(i) $\sin = -\frac{3}{5}$
$\cos = \pm\frac{4}{5}$
$\tan = \mp\frac{3}{4}$

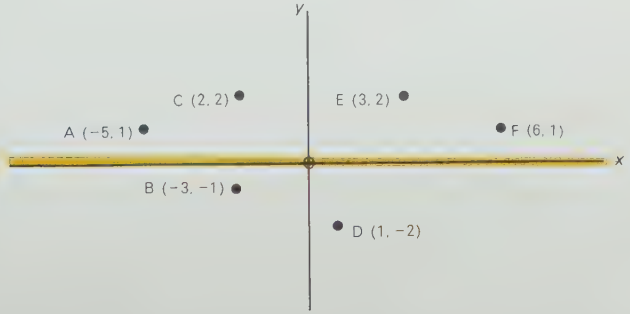
16.4 Trig Values of Angles Greater Than 90°

A If we use the x axis as the mirror, we can find the reflection image of a point using coordinates.



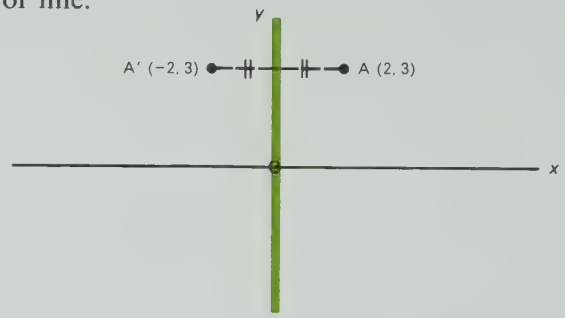
What do you notice about the distances A to the x axis and A′ to the x axis?

1. State the coordinates of the image of each of the following points (x axis is the mirror).



Images of B and D are above the x axis.

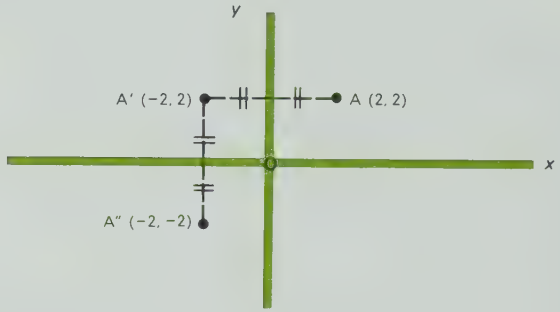
We can also find the reflection image of a point using the y axis as the mirror line.



The image of A(2, 3) is A′ (−2, 3)

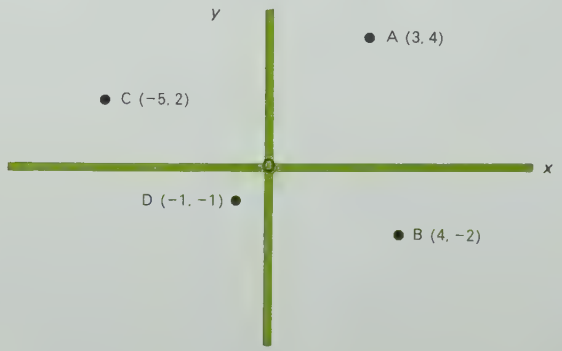
Compare the distances A to the y axis and A′ to the y axis.

We can also have double reflections, that is, we can reflect a point in *both* axes.



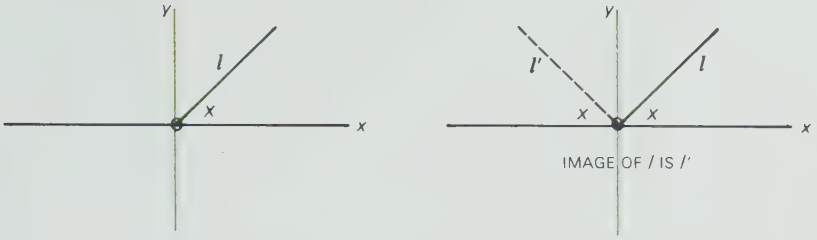
The double reflection image of A(2, 2) is A″(−2, −2)

2. State the coordinates of the image of each of the following points after a reflection in both axes.



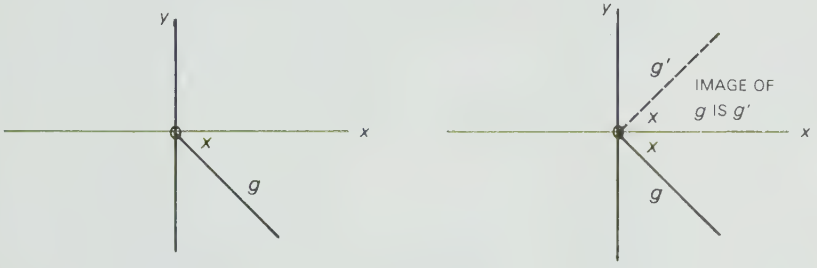
B Let's find the images of lines,

(*a*) using the y axis as the mirror,

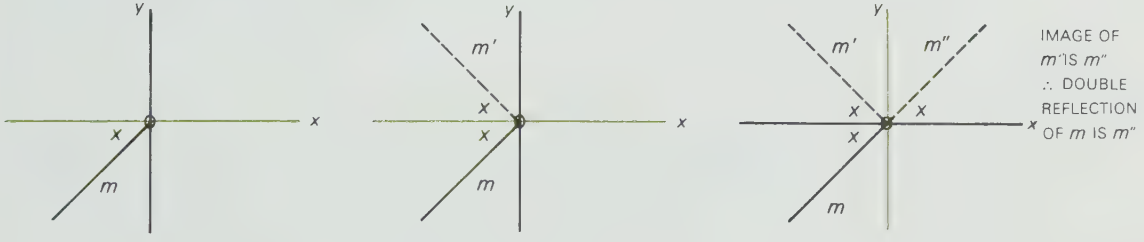


Notice which angles are equal.

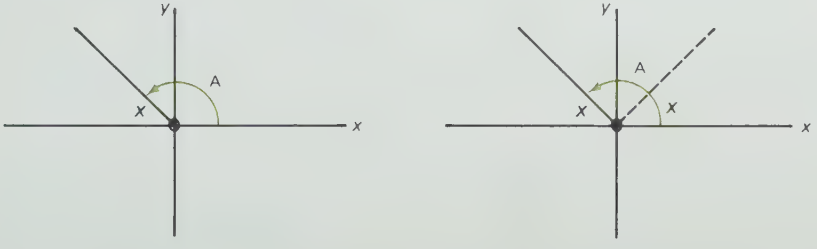
(*b*) using the x axis as the mirror,



(*c*) using both axes as mirror lines.



We can use the concept of reflections to find the reflection image in the first quadrant of an angle whose terminal arm is in any other quadrant.



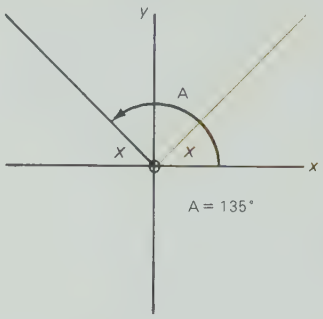
If A = 120°
x = 60°

The reflection of the terminal arm of A gives us a 60° angle in the 1st quadrant.

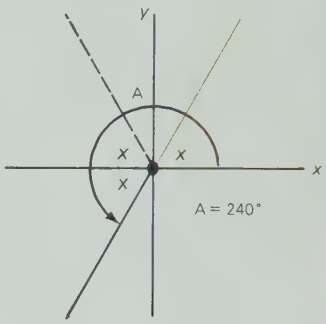
x is called the *related acute angle.*

1. Calculate the value of the related acute angle x in each of the following diagrams, where the colored line is the image of the black line.

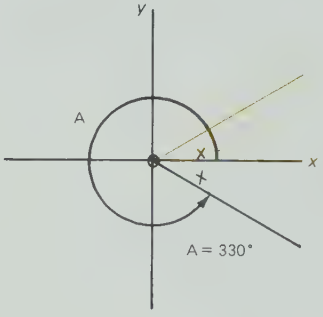
(*a*)



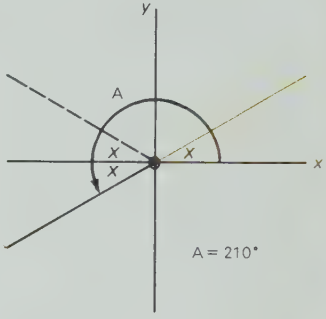
(*c*)



(*b*)



(*d*)

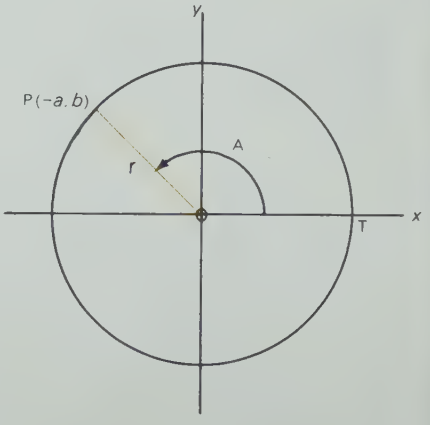


Two reflections for angles in the third quadrant.

C We will now show how to find trig values of 2nd-quadrant angles using the idea of reflections.

Let $\angle TOP = A$ be an angle in the second quadrant.
$\therefore$ A is greater than 90° but less than 180°.

Let $P(-a, b)$ be a point on the terminal arm and let $OP = r$.



$$\sin A = \frac{b}{r}$$

$$\cos A = \frac{-a}{r} = -\frac{a}{r}$$

$$\tan A = \frac{b}{-a} = -\frac{b}{a}$$

Find the reflection image of OP that falls in the 1st quadrant.
Reflect in the y axis.
Let the image of P be P′.
$\therefore \angle \text{TOP}' = x = 180° - \text{A}$.
Notice that

$$\sin x = \frac{b}{r}$$

$$\cos x = \frac{a}{r}$$

$$\tan x = \frac{b}{a}$$

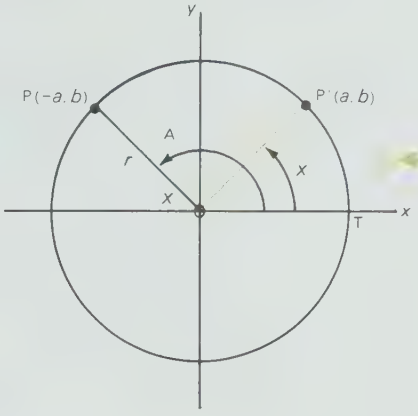
$$\therefore \sin \text{A} = \frac{b}{r} = \sin x$$

$$\cos \text{A} = -\frac{a}{r} = -\cos x$$

$$\tan \text{A} = -\frac{b}{a} = -\tan x$$

where $90° < \text{A} < 180°$ and $0° < x < 90°$.

$\therefore$ we can find trig values of angles between 90° and 180° using trig values of angles between 0° and 90°.



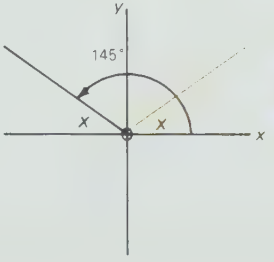
Notice the related acute angle.

Example

Find the trig values of an angle of 145° in terms of angles between 0° and 90°.

Solution

$\text{A} = 145°$
$\therefore x = 180° - 145° = 35°$
$\therefore \sin \text{A} = \sin 145° = \sin 35°$
$\cos \text{A} = \cos 145° = -\cos 35°$
$\tan \text{A} = \tan 145° = -\tan 35°$

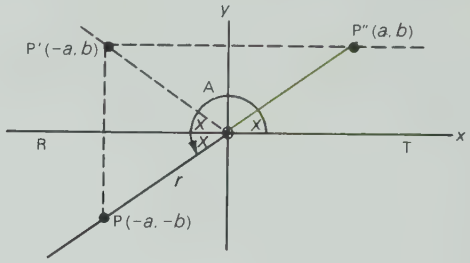


Look for the related acute angle.

Recall the CAST rule.

1. Find the trig values of the following angles in terms of angles between 0° and 90°.
   (*a*) 135°
   (*b*) 120°
   (*c*) 153°
   (*d*) 160°
   (*e*) −250°
   (*f*) 172°

D Angles in the third quadrant can be reflected in a similar manner. Let A be a third-quadrant angle and P a point on the terminal arm. Two reflections are required to obtain an image in the first quadrant.



If P has coordinates $(-a, -b)$, its image P″ has coordinates (a, b) as shown in the diagram.

The related acute angle.

Let $\angle ROP = x$ where $A = 180° + x$ and $0° < x < 90°$.

$\therefore \angle TOP'' = x$

Write the trig values of A and x.

$$\sin A = \frac{-b}{r} = -\frac{b}{r} \qquad \sin x = \frac{b}{r}$$

$$\cos A = \frac{-a}{r} = -\frac{a}{r} \qquad \cos x = \frac{a}{r}$$

$$\tan A = \frac{-b}{-a} = \frac{b}{a} \qquad \tan x = \frac{b}{a}$$

Comparing the two sets of values we get

$$\sin A = -\sin x$$
$$\cos A = -\cos x$$
$$\tan A = \tan x$$

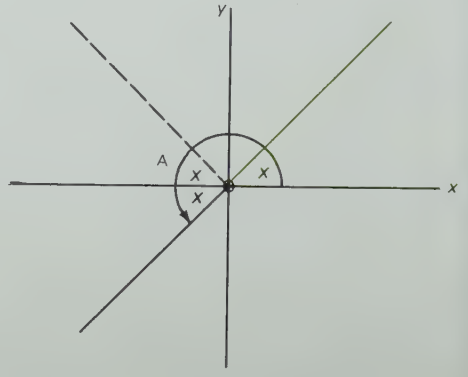
where A is an angle between 180° and 270° and $x = A - 180°$.

Example

Find the trig values of an angle of 202° in terms of angles between 0° and 90°.

Solution

$$A = 202°$$
$$\therefore x = 202° - 180° = 22°$$
$$\therefore \sin A = \sin 202°$$
$$= -\sin 22°$$
$$\cos A = \cos 202°$$
$$= -\cos 22°$$
$$\tan A = \tan 202°$$
$$= \tan 22°$$

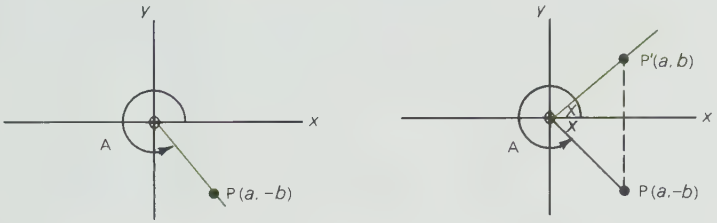


1. Express the three trig values of the following angles in terms of angles between 0° and 90°.

(*a*) 240°	(*c*) 258°	(*e*) 190°
(*b*) 187°	(*d*) 265°	(*f*) −110°

E The trigonometric values of 4th-quadrant angles can also be related to 1st-quadrant angles.

Step 1 Get image line. Find x.



Step 2 Find trigonometric values of A and x (the related acute angle).

$$\sin A = \frac{-b}{r} = -\frac{b}{r}, \quad \sin x = \frac{b}{r}$$

$$\cos A = \frac{a}{r} \qquad \cos x = \frac{a}{r}$$

$$\tan A = \frac{-b}{a} = -\frac{b}{a} \quad \tan x = \frac{b}{a}$$

Check ± sign, using "CAST".

Step 3 Compare.

$$\sin A = -\sin x$$
$$\cos A = \cos x$$
$$\tan A = -\tan x$$

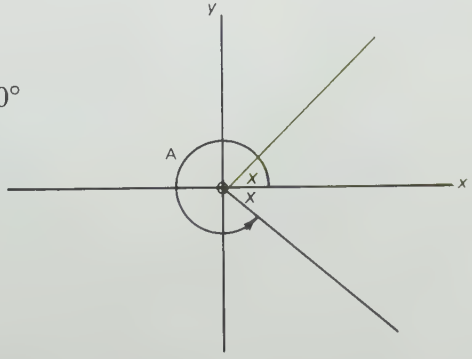
where $270° < A < 360°$ and $x = 360° - A$

Example

Find the related 1st-quadrant trig values of an angle of 300°.

Solution

$$A = 300°$$
$$\therefore x = 360° - 300° = 60°$$
$$\therefore \sin A = \sin 300°$$
$$= -\sin 60°$$
$$\cos A = \cos 300°$$
$$= \cos 60°$$
$$\tan A = \tan 300°$$
$$= -\tan 60°$$



CAST

1. Find the related 1st-quadrant trig values of the following angles.
   (*a*) 302° (*c*) 354° (*e*) 330°
   (*b*) 280° (*d*) 311° (*f*) −70°

Here is a summary of results.

sin A = sin x cos A = −cos x tan A = −tan x	
sin A = −sin x cos A = −cos x tan A = tan x	sin A = −sin x cos A = cos x tan A = −tan x

Notice that the positive results give the CAST rule again.

*S*ine positive	*A*ll positive	S	A
*T*angent positive	*C*osine positive	T	C

or

Thus, to find the trig value of *any* angle, follow these steps.

1. Draw a diagram and determine the quadrant in which the terminal arm lies.
2. Find the related acute angle between the terminal arm and the x axis.
3. Find the trig value of the acute angle.
4. Use the CAST rule to determine the sign of the answer.

Exercise 16.4

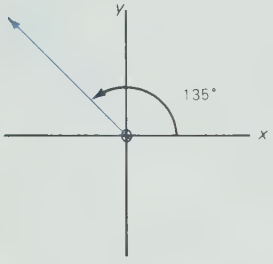
Discussion

1. State the images of the following points using the x axis as a mirror line. Repeat using the y axis as a mirror line.
   (*a*) (−8, 2) (*f*) (−8, 9)
   (*b*) (5, −6) (*g*) (5, −5)
   (*c*) (−2, −7) (*h*) (−3, 9)
   (*d*) (5, −1) (*i*) $(-\sqrt{5}, -\sqrt{3})$
   (*e*) (−7, −3) (*j*) $(\sqrt{2}, -6)$

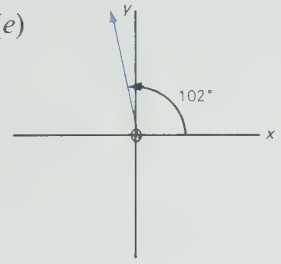
Written Solutions

2. Reflect in either the x axis, y axis, or both axes so that the image of the terminal arm of the following angles is in the first quadrant. Sketch diagrams and mark all equal angles.

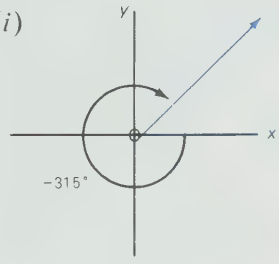
(a)



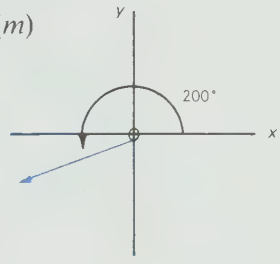
(e)



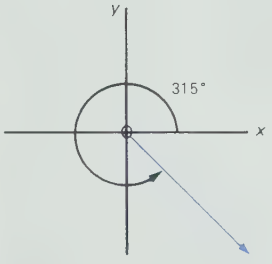
(i)



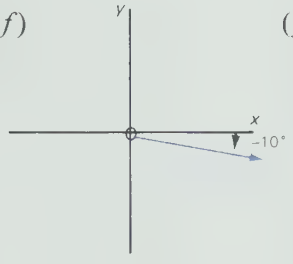
(m)



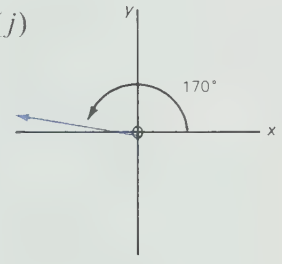
(b)



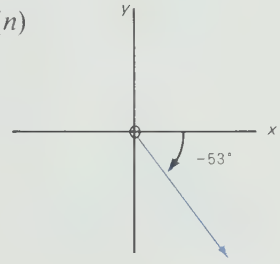
(f)



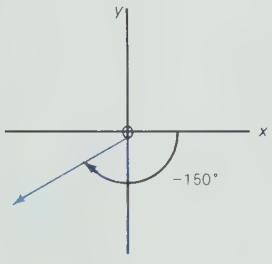
(j)



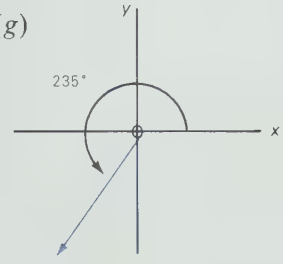
(n)



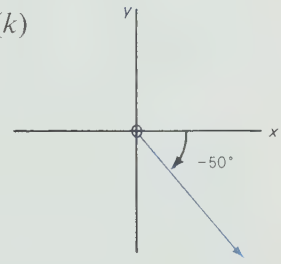
(c)



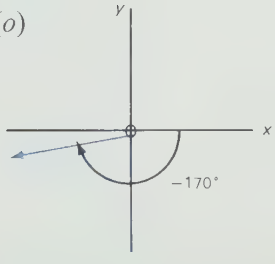
(g)



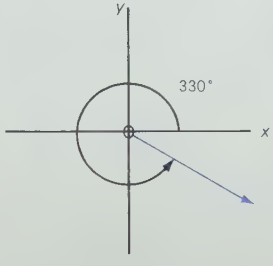
(k)



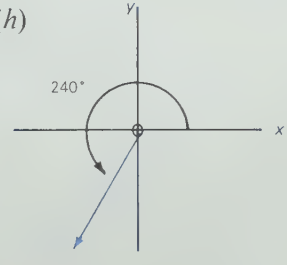
(o)



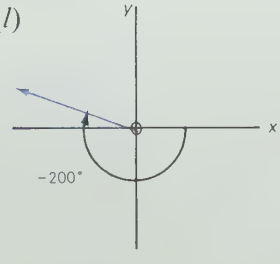
(d)



(h)



(l)



3. Express each of the following in terms of the related acute angle.

(*a*) sin 110°	(*e*) sin 260°
(*b*) sin 175°	(*f*) sin 280°
(*c*) sin 197°	(*g*) sin 300°
(*d*) sin 230°	(*h*) sin 350°

4. Express each of the following in terms of the related acute angle.

(*a*) cos 112°	(*e*) cos 265°
(*b*) cos 170°	(*f*) cos 285°
(*c*) cos 200°	(*g*) cos 302°
(*d*) cos 250°	(*h*) cos 346°

5. Express each of the following in terms of the related acute angles.

(*a*) tan 96°	(*e*) tan 282°
(*b*) tan 117°	(*f*) tan 300°
(*c*) tan 205°	(*g*) tan 305°
(*d*) tan 260°	(*h*) tan 352°

3.(a) sin 70°
(b) sin 5°
(e) − sin 80°
(f) − sin 80°

4.(a) − cos 68°
(b) − cos 10°
(e) − cos 85°
(f) cos 75°

5.(a) − tan 84°
(b) − tan 63°
(e) − tan 78°
(f) − tan 60°

Use the tables pages 462-463

16.5 Trig Values of Any Angle Using Tables

The tables can now be used to find the trig values of any angle by finding the related acute angle and then applying the CAST rule to determine the sign.

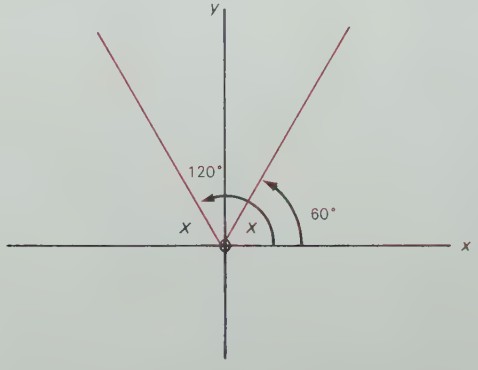
Example 1

Find the trig values of 120°.

Solution

Steps 1 and 2

Determine the corresponding acute angle in the 1st quadrant.



$x = 180° - 120° = 60°$
$\therefore$ related acute angle is 60°

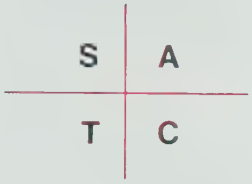
Step 3

Find the trig values of this acute angle.

$$\sin 60° = 0.8660$$
$$\cos 60° = 0.5000$$
$$\tan 60° = 1.7321$$

Step 4

Determine the correct sign.



Since 120° is a 2nd-quadrant angle, only the sine value is positive.

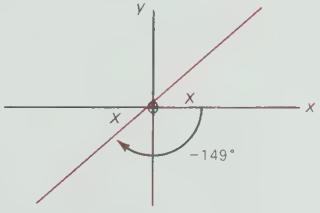
$$\therefore \sin 120° = \sin 60° = 0.8660$$
$$\cos 120° = -\cos 60° = -0.5000$$
$$\tan 120° = -\tan 60° = -1.7321$$

Example 2

Find the trig values for $-149°$.

Solution

Step 1

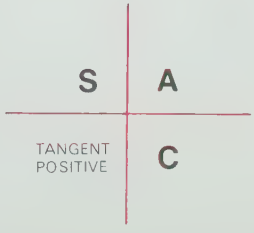


Step 2 The related acute angle is $180° - 149° = 31°$.

Step 3
$$\sin 31° = 0.5150$$
$$\cos 31° = 0.8572$$
$$\tan 31° = 0.6009$$

Step 4 Apply the CAST rule.

$$\sin(-149°) = -0.5150$$
$$\cos(-149°) = -0.8572$$
$$\tan(-149°) = 0.6009$$



1. Find the trig values of each of the following angles.

(*a*) 98° (*c*) 351° (*e*) −133°
(*b*) 260° (*d*) −40° (*f*) −215°

Use diagrams!

See the tables pages 462-463

Exercise 16.5

Written Solutions

1. Determine the value of each of the following. Sketch a diagram in each case.
   (*a*) sin 153°
   (*b*) cos 320°
   (*c*) tan 217°
   (*d*) tan 117°
   (*e*) cos 235°
   (*f*) sin 240°
   (*g*) sin 330°
   (*h*) cos 127°
   (*i*) tan 356°
   (*j*) sin 210°
   (*k*) cos 160°
   (*l*) tan 265°
   (*m*) sin (−10°)
   (*n*) cos (−100°)

Work backward from the tables.

2. Each of the following is the sine of two angles between 0° and 360°. Find the angles and sketch a diagram to illustrate.
   (*a*) 0.616
   (*b*) −0.309
   (*c*) 0.927
   (*d*) 0.087
   (*e*) −0.707
   (*f*) −0.990
   (*g*) 0.191
   (*h*) 0.766

3. Determine two angles between 0° and 360° each of whose cosine is the following.
   (*a*) 0.643
   (*b*) −0.342
   (*c*) −0.035
   (*d*) 0.940
   (*e*) 0.996
   (*f*) 0.018
   (*g*) −0.857
   (*h*) −0.695

4. Determine two angles between 0° and 360° each of whose tangent is the following.
   (*a*) 0.404
   (*b*) 0.810
   (*c*) 11.430
   (*d*) −8.144
   (*e*) −0.287
   (*f*) 1.192
   (*g*) −2.050
   (*h*) 0.141

1.(b) 0.7660
(c) 0.7536
(f) −0.8660
(g) −0.5000
(i) −0.0699
(j) −0.5000
(m) −0.1736
(n) −0.1736

2.(a) 38°, 142°
(b) 198°, 342°
(e) 225°, 315°
(f) 262°, 278°

3.(b) 110°, 250°
(c) 92°, 268°
(f) 89°, 271°
(g) 149°, 211°

4.(c) 85°, 265°
(d) 97°, 277°
(g) 116°, 296°
(h) 8°, 188°

A Geometric Contradiction

Experiment 1

1. Take a map for a few square kilometres of the earth's surface. Now place two pencils pointing north and about 2 cm apart. Move each pencil due north. Are the pencils still 2 cm apart?
2. Place the two pencils parallel and pointing east. Move each pencil due east. Would one pencil still be due north of the other?
3. Mark a point on the map. To get to another point that is due east of the pencil mark, is it best to move the pencil point due east?

You probably answered all these questions with a yes. In fact you probably felt that all these questions were silly and obvious.

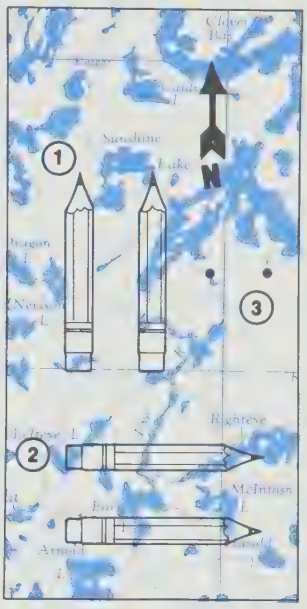
However, to envision what happens on a much larger scale let's try these ideas on a globe. Borrow one from your geography department and try the same ideas again.

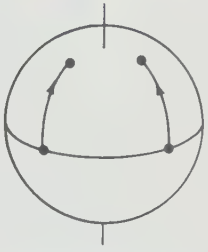
Experiment 2

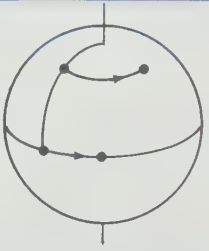
1. Place two pencil marks on the equator 5 cm apart. Now locate each point the same distance due north. Do they remain 5 cm apart?
2. Place two marks on the globe, one at the equator and the other due north of it. Move each point the same distance due east. Is the one point still due north of the other?
3. Place a mark somewhere near the north pole. To get to a point due east of the other, is it best to move due east?

The answers to these questions about the globe, which are exactly the same questions as those asked about the map, are all no.

When we think about it, it is misleading to represent a curved surface such as the earth by a flat map. Measurements from a flat surface are not the same as those from a curved surface. For small portions of the earth, the curve is so slight it is not noticeable. This is the reason geometry "works" — the errors are so small that the results are accurate enough to be satisfactory. This is why we still study Greek geometry, which is usually called Euclidean geometry after the Greek mathematician Euclid.







Unit 17

Graphing and Solution of Oblique Triangles

An OBLIQUE Triangle is any triangle that does not contain a right angle.

17.1 Trigonometric Values of 0°, 90°, 180°, 270°, and 360°

We can use the idea of an angle in standard position to find the trig values of some very special angles.

(*a*) What are the trig values of 0°?

When $\theta = 0°$

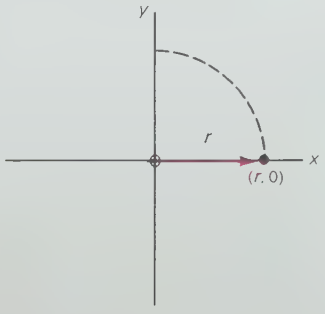
$y = 0$

and $x = r$.

$\therefore \sin \theta = \dfrac{0}{r} = 0$

$\cos \theta = \dfrac{x}{r} = 1$

$\tan \theta = \dfrac{0}{x} = 0$



$\sin \theta = \frac{y}{r}$

$\cos \theta = \frac{x}{r}$

$\tan \theta = \frac{y}{x}$

You can not divide by zero

r is always considered to be positive

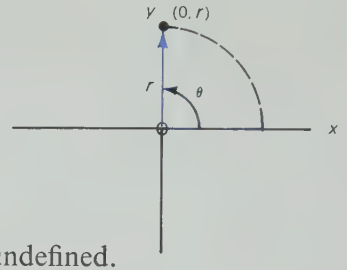
(*b*) When $\theta = 90°$

$x = 0$ and $y = r$.

$\therefore \sin 90° = \dfrac{y}{r} = 1$

$\cos 90° = \dfrac{0}{r} = 0$

$\tan 90° = \dfrac{y}{x} = \dfrac{y}{0}$, which is undefined.



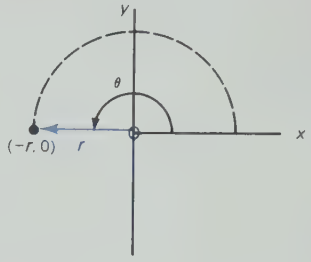
(*c*) When $\theta = 180°$

$y = 0$ and $x = -r$.

$\therefore \sin 180° = \dfrac{y}{r} = 0$

$\cos 180° = \dfrac{x}{r} = -1$

$\tan 180° = \dfrac{0}{r} = 0$



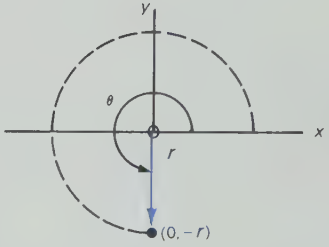
(*d*) When $\theta = 270°$

$x = 0$ and $y = -r$.

$\therefore \sin 270° = -1$

$\cos 270° = 0$

$\tan 270°$ is undefined. Why?

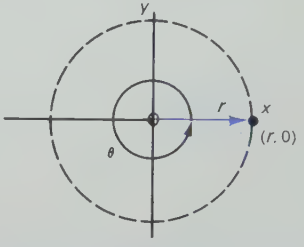


(*e*) Since angles of 0° and 360° are coterminal, that is, the final arms are in the same position, then the trig values of 0° and 360° are the same.

$\sin 360° = 0$

$\cos 360° = 1$

$\tan 360° = 0$



Summary

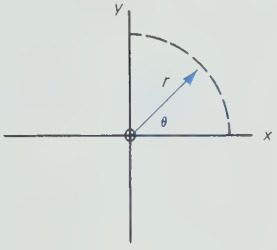
θ	0°	90°	180°	270°	360°
$\sin\theta$	0	1	0	1	0
$\cos\theta$	1	0	1	0	1
$\tan\theta$	0	undefined	0	undefined	0

1. Evaluate.
   (*a*) $\sin 0° + 2\cos 360° \times \sin 270° + 3\sin 90°$
   (*b*) $\cos 0° \times \sin 90° \times \cos 180° \times \sin 270° + 3\cos 90°$
   (*c*) $\left(\sin\frac{\pi}{2}\right)^2 + \left(\cos\frac{\pi}{2}\right)^2 - 2\cos\pi + 3\sin\frac{3\pi}{2}$
   (*d*) $\cos 0 \times \cos\pi + \sin\frac{\pi}{2} \times \sin\frac{3\pi}{2} + 2\cos\frac{\pi}{2} \times \cos 2\pi$

Recall that π rad = 180°

Exercise 17.1

Discussion

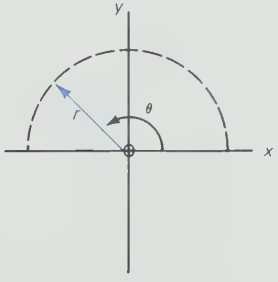
1. 

As θ increases from 0° to 90°, describe the change in each of the following: x, y, r, $\sin\theta$, and $\cos\theta$.

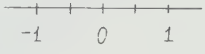
$\sin\theta = \frac{y}{r}$

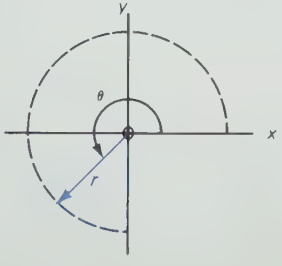
$\cos\theta = \frac{x}{r}$

2. As θ increases from 90° to 180°, describe the change in each of the following: x, y, r, $\sin\theta$, and $\cos\theta$.



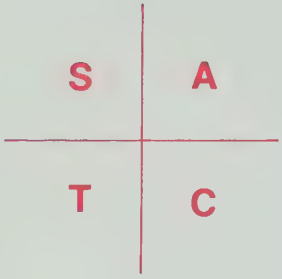
Which is greater, -1 or $-\frac{1}{2}$?



3. 

As θ continues to increase from 180° to 270°, what is happening to the values of x, y, r, $\sin\theta$, and $\cos\theta$?

4. What happens to the values of x, y, r, $\sin\theta$, and $\cos\theta$ as θ increases from 270° to 360°?



Written Solutions

5. Copy and complete the following table.

	x	y	$\sin\theta$	$\cos\theta$
1st quadrant	+	+	+ inc.	+ dec.
2nd quadrant	−			dec.
3rd quadrant				
4th quadrant				

6. Evaluate.
   (*a*) $\cos 0° \sin^2 270° - 2 \cos 180° \tan 45°$
   (*b*) $3 \sin 0° \cos 180° + 2 \sin 90° - \cos 360°$
   (*c*) $2 \cos^2 \pi \sin 0° + 3 \sin^3 \frac{3\pi}{2} - \cos \frac{\pi}{2}$
   (*d*) $\tan \pi \cos \frac{3\pi}{2} + \cos 2\pi - \sin \frac{3\pi}{2}$

7. State the maximum and minimum values of the following as θ rotates through 360°.
   (*a*) $\sin\theta$
   (*b*) $\cos\theta$
   (*c*) $\tan\theta$
   (*d*) $\sin^2\theta$
   (*e*) $\cos 4\theta$
   (*f*) $\tan \frac{3\theta}{2}$

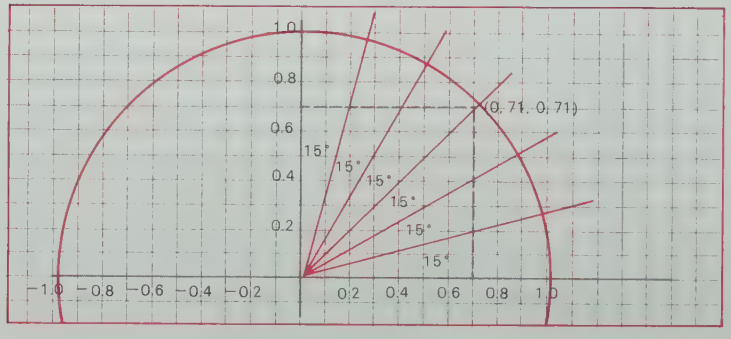
6.(a) 3
(b) 1

	Max.	Min.
7.(a) $\sin\theta$	1	−1
(d) $\sin^2\theta$	1	0

17.2 The Graphs of $\sin\theta$, $\cos\theta$, and $\tan\theta$

Project Exercise

1. A circle with radius 1 unit has its centre at the origin of the coordinate axes.
   (*a*) Using points on the circumference of the circle and the terminal arms of angles at 15° intervals, find the corresponding values of x and y.



$\sin\theta = \frac{y}{r}$
$\cos\theta = \frac{x}{r}$
If $\theta = 45°$ and
$r = 1$, then
$\sin 45° = 0.71$ and
$\cos 45° = 0.71$

(*b*) Copy and complete the table.

	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	... 360°
x												
y												
r												
sin θ												
cos θ												
tan θ												

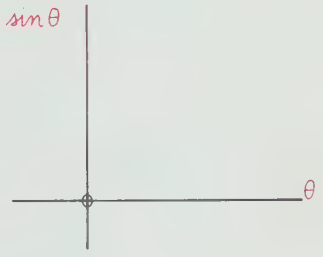
2. *To graph sin* θ

(*a*) Label the coordinate axes θ and sin θ and, by forming ordered pairs $(\theta, \sin \theta)$, plot points on the graph. Sketch the curve.

(*b*) What is the maximum value of sin θ?

(*c*) What is the minimum value of sin θ?

(*d*) What happens to the curve when $\theta > 360°$? $\theta < 0°$?



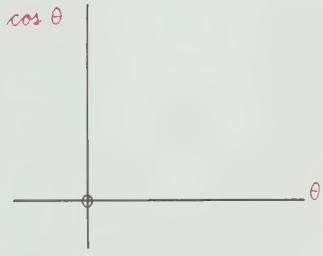
3. *To graph cos* θ

(*a*) Label the coordinate axes θ and cos θ and plot the points from the table in Question 1. Sketch the curve.

(*b*) What is the maximum value of cos θ?

(*c*) What is the minimum value of cos θ?

(*d*) What happens to the curve when $\theta > 360°$? $\theta < 0°$?



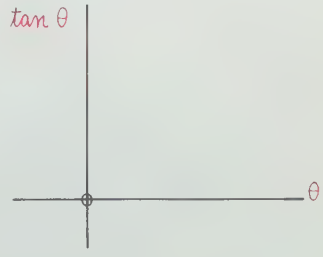
4. *To graph tan* θ

(*a*) Label the coordinate axes θ and tan θ.

(*b*) At the values of θ for which tan θ is undefined, draw vertical lines since no point on the graph can lie on these lines.

(*c*) Plot the values of θ and tan θ from the table in Question 1 and sketch the curve.

(*d*) What happens to the curve when $\theta > 360°$? $\theta < 0°$?



5. Draw the graphs of the following for $-360° \leqslant \theta \leqslant 360°$.

(*a*) $-\sin \theta$

(*b*) $2 \sin \theta$

(*c*) $-\cos \theta$

(*d*) $\frac{1}{2} \cos \theta$

(*e*) $3 \cos \theta$

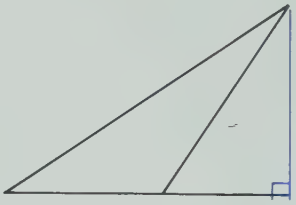
(*f*) $-3 \sin \theta$

(*g*) $-2 \cos \theta$

(*h*) $\frac{1}{2} \sin \theta$

17.3 The Law of Sines

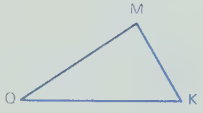
Any acute-angled or obtuse-angled triangle could be solved by drawing one of its altitudes, and applying the methods we used for right-angled triangles.

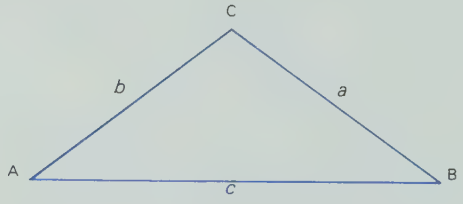


However, for efficiency and convenience the solution of oblique triangles is accomplished by the use of special formulas. One such formula is known as the *Law of Sines*.

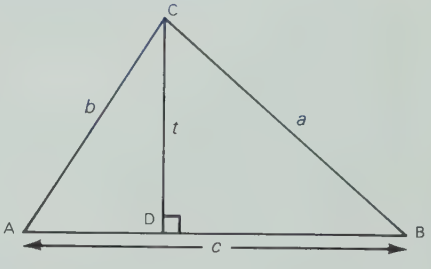
In $\triangle ABC$, side a is opposite $\angle A$, side b is opposite $\angle B$, and side c is opposite $\angle C$.

How would you label the sides of $\triangle MQK$?





In $\triangle ABC$, draw the altitude CD.
Let $AC = b$, $CB = a$, $CD = t$, and $AB = c$.



$\therefore \sin A = \frac{t}{b}$, $\sin B = \frac{t}{a}$

$\therefore b \sin A = t$ and $a \sin B = t$

$\therefore b \sin A = a \sin B$

$\therefore \frac{a}{\sin A} = \frac{b}{\sin B}$ (divide each side by $\sin A \times \sin B$)

This is the first part of the Law of Sines. Which perpendicular would we use to get an equation involving side c and $\angle C$?

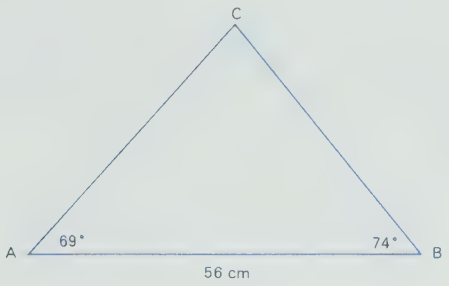
The Law of Sines

> In any triangle, the sides are proportional to the sines of the opposite angles.
>
> $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 1

Solve $\triangle ABC$ if $\angle A = 69°$, $\angle B = 74°$, and $c = 56$ cm.

Solution



To solve a $\triangle$ we must find
— the lengths of the three sides
— the size of the three angles.

Since two angles are known, we can find the third.

$$\begin{aligned}\angle C &= 180° - (69° + 74°)\\ &= 180° - 143°\\ &= 37°\end{aligned}$$

From the sine law,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

From $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

we get $b = \dfrac{c \sin B}{\sin C}$

$$= \frac{(56)(0.9613)}{0.6018}$$

$$\doteq 89$$

$\therefore b$ is approximately 89 cm.

Similarly,

from $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

we get $a = \dfrac{c \sin A}{\sin C}$

$$= \frac{(56)(0.9336)}{0.6018}$$

$$\doteq 87$$

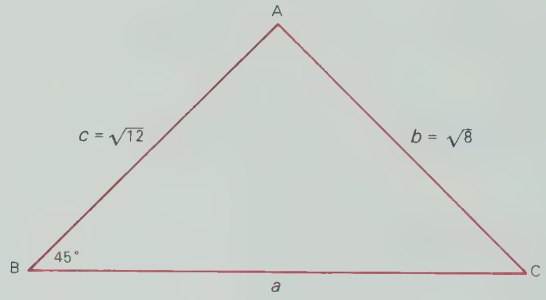
$\therefore a$ is approximately 87 cm.

Example 2

Solve $\triangle ABC$ if $\angle B = 45°$, $c = \sqrt{12}$ and $b = \sqrt{8}$.

Solution

First make a diagram and mark on it the angles and sides you know.



To solve $\triangle ABC$ we must find side a, $\angle A$, and $\angle C$.

From the sine law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

To determine sin C,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore b \sin C = c \sin B$$

$$\sin C = \frac{c \times \sin B}{b}$$

$$= \frac{\sqrt{12}}{\sqrt{8}} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{12}}{\sqrt{16}}$$

$$= \frac{2\sqrt{3}}{4}$$

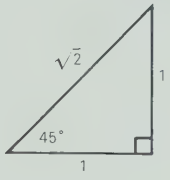
$$= \frac{\sqrt{3}}{2}$$

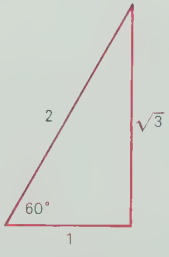
$$\text{If } \sin C = \frac{\sqrt{3}}{2}, \quad \therefore C = 60° \text{ or } 120°.$$

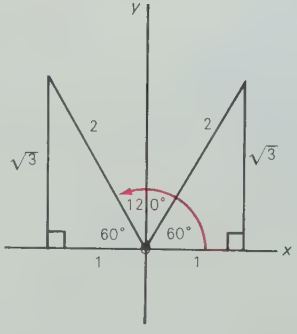
To determine $\angle A$,

If $C = 60°$,
then $A = 180° - 45° - 60°$
$= 75°$.

If $C = 120°$,
then $A = 180° - 45° - 120°$
$= 15°$.







To determine side a, *two* values will be required, since $\angle A$ can have two values.

If $A = 75°$

since $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$$\therefore a = \frac{b \times \sin A}{\sin B}$$

$$= \frac{\sqrt{8} \times \sin 75°}{\frac{1}{\sqrt{2}}}$$

$$\doteq \sqrt{8} \times 0.966 \times \frac{\sqrt{2}}{1}$$

$$= \sqrt{16} \times 0.966$$

$$= 4 \times 0.966$$

$$= 3.864$$

$$\doteq 3.9$$

If $A = 15°$

$$\therefore a = \frac{b \times \sin A}{\sin B}$$

$$= \frac{\sqrt{8} \times \sin 15°}{\frac{1}{\sqrt{2}}}$$

$$\doteq \sqrt{8} \times 0.259 \times \frac{\sqrt{2}}{1}$$

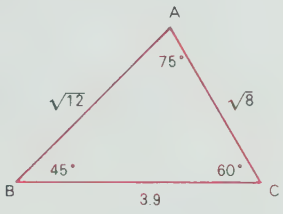
$$= \sqrt{16} \times 0.259$$

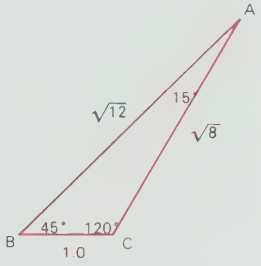
$$= 4 \times 0.259$$

$$= 1.036$$

$$\doteq 1.0$$

We have solved the triangle! And there are *two* possible solutions.





Exercise 17.3

Written Solutions

1. In each of the following, solve $\triangle ABC$. Find angles to the nearest degree and sides correct to two digits.
   (*a*) $\angle A = 127°$, $\angle B = 43°$, $a = 66$ m
   (*b*) $\angle A = 66°$, $\angle B = 36°$, $c = 8.4$ cm
   (*c*) $\angle B = 40°$, $\angle C = 25°$, $a = 24$ cm
   (*d*) $\angle A = 25°$, $b = 125$ cm, $a = 80$ cm
   (*e*) $c = 79$, $\angle A = 46°$, $\angle B = 29°$
   (*f*) $c = 10$, $a = 14$, $\angle A = 73°$
   (*g*) $b = 25$, $c = 20$, $\angle C = 38°$
   (*h*) $b = 15$, $\angle C = 28°$, $\angle A = 52°$
   (*i*) $\angle A = 123°$, $a = 4.3$, $b = 2.5$
   (*j*) $\angle A = 84°$, $\angle B = 62°$, $b = 2$

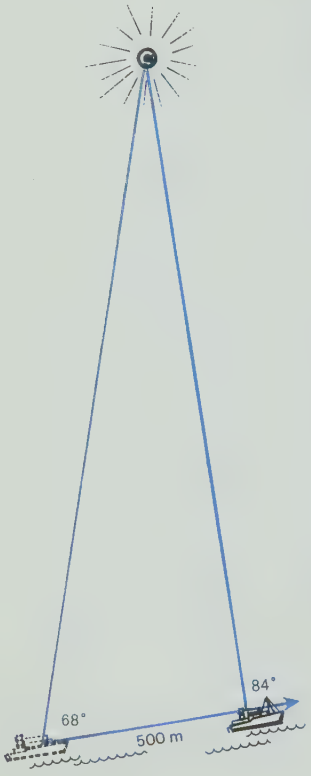
To solve a △ means to find: —the lengths of the 3 sides
—the sizes of the 3 angles

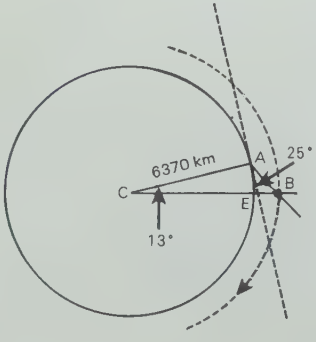
1.(a) $\angle C = 10°$, $b = 56$ m, $c = 14$ m
(b) $\angle C = 78°$, $a = 7.8$ cm, $b = 5.0$ cm
(h) $a = 12$, $c = 7.2$, $\angle B = 100°$
(i) $\angle B = 29°$, $\angle C = 28°$, $c = 2.4$

3. 950 m from A and 1340 m from B

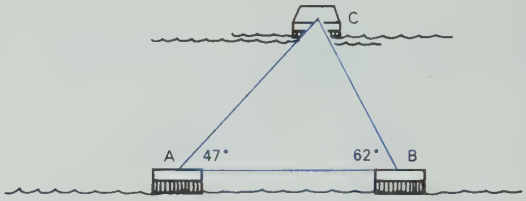
5. 268 m

7. $x \doteq 6.71$

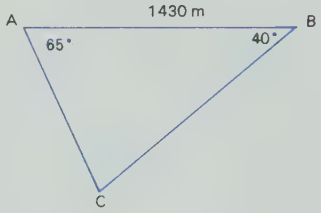




2. Two docks A and B on shore are 1.7 km apart. How far is an island dock C from A and B if $\angle BAC = 47°$ and $\angle CBA = 62°$?

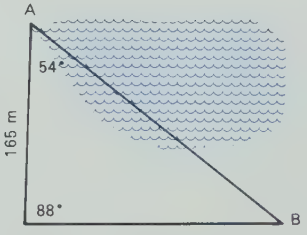


3. Two observation posts A and B are 1430 m apart. How far is a tower C from A and B if $\angle ABC = 40°$ and $\angle CAB = 65°$?



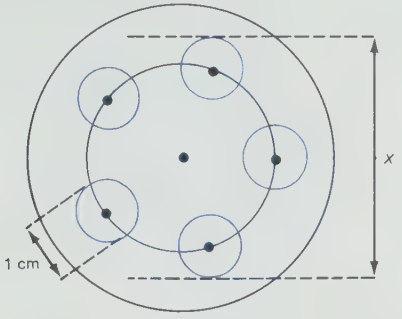
4. A navigator on a boat measures the angle between the course of the ship and a lighthouse on shore to be 68°. After the boat has travelled 500 m, he measures the angle to be 84°. How far was the boat from the lighthouse at the time of the first sighting and at the second sighting?

5. Find the inaccessible distance AB across a pond from the following survey data.



6. At point E on the surface of the earth, a satellite, B, is observed directly overhead. At that instant, the angle of elevation of B from a tracking station A is 25°. If A is on the same line of longitude but has latitude 13° different from E, find the altitude EB of the satellite to the nearest kilometre.

7. Find x in the following diagram. The five holes in the circle of 3 cm radius are evenly spaced.

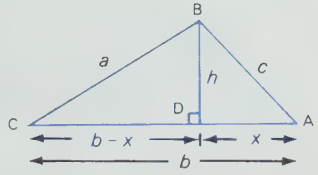


8. A horizontal bridge 78 m long is built across a ravine. From a certain point, A, at the bottom of the ravine and directly under the bridge a pier is erected to support the bridge. If the angles of elevation of the two ends of the bridge as seen from the point A are 57° and 68°, find the height of the required pier.

17.4 The Law of Cosines

Another formula that will help in the solution of oblique triangles is called the *Law of Cosines.*

Given $\triangle ABC$ where A, b, and c are known and where A is an acute angle, we can find a formula for a in terms of A, b and c.



Let h be the altitude from B to side b.

$$\therefore \frac{x}{c} = \cos A$$

$$\therefore x = c \times \cos A$$

Using the Pythagorean Theorem on $\triangle BCD$ and $\triangle BDA$,

$$a^2 = h^2 + (b - x)^2 \qquad (1)$$

$$\text{and } c^2 = h^2 + x^2$$

$$h^2 = c^2 - x^2$$

By substitution in (1)

$$\begin{aligned} a^2 &= (c^2 - x^2) + (b - x)^2 \\ &= c^2 - x^2 + b^2 - 2bx + x^2 \\ &= c^2 + b^2 - 2bx \end{aligned}$$

$$a^2 = c^2 + b^2 - 2bc \cos A$$

This is known as the Law of Cosines.

Since $x = c \times \cos A$,
$\therefore 2bx = 2bc \times \cos A$

What a mouthful! Bet it's easier to memorize the formulas!

The Law of Cosines

In any triangle the square on one side is equal to the sum of the squares on the other two sides minus twice the product of these sides times the cosine of the angle between them.

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A$$
$$\text{or } b^2 = a^2 + c^2 - 2ac \cos B$$
$$\text{or } c^2 = a^2 + b^2 - 2ab \cos C$$

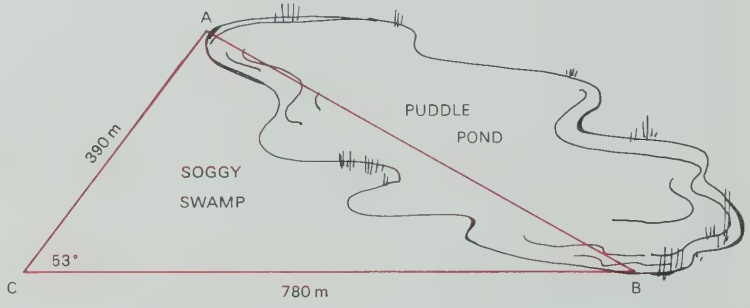
In another form: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$$\text{or } \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{or } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 1

The inaccessible distance AB across a pond is to be found from the following survey data.



Solution

In $\triangle ABC$, AB $= c$, $b = 390$, $a = 780$, C $= 53°$.

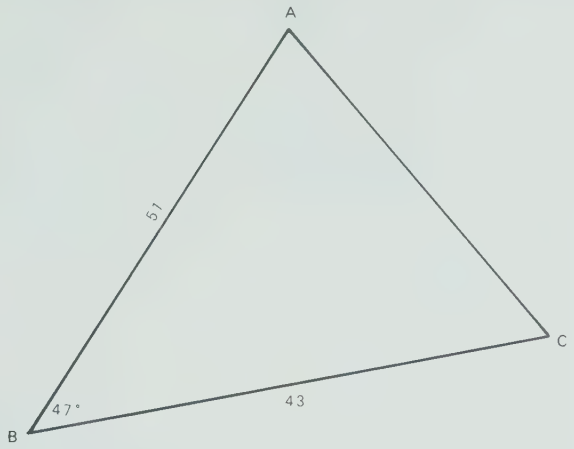
$$\therefore c^2 = a^2 + b^2 - 2ab \cos C$$
$$= (780)^2 + (390)^2 - 2(780)(390) \cos 53°$$
$$\doteq 610\,000 + 152\,000 - 608\,400(0.602)$$
$$\doteq 762\,000 - 365\,000$$
$$= 397\,000$$
$$\therefore c \doteq 630$$

$\therefore$ the distance across the pond is approximately 630 m.

$\sqrt{397000}$
$= \sqrt{3970 \times 100}$
$= 10\sqrt{3970}$
$\doteq 10 \times 63$
$= 630$

Example 2

Solve the triangle ABC, given that $c = 51$, $a = 43$, $B = 47°$.



Solution

Find b first using the formula

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B \\ &= (43)^2 + (51)^2 - 2(43)(51) \cos 47° \\ &= 1849 + 2601 - 4386(0.682) \\ &= 4450 - 2991 \\ &= 1459 \end{aligned}$$

$$\therefore b \doteq 38$$

Now that we have a, b, and c, we can find $\angle A$ using the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{1459 + 2601 - 1849}{2 \times 38 \times 51}$$

$$= \frac{2211}{3876}$$

$$\doteq 0.570$$

$$\therefore A \doteq 55°$$

$$\therefore C = 180° - 47° - 55°$$

$$= 78°$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{43} = \frac{\sin 47°}{38}$$

$$\therefore \sin A \doteq \frac{43 \times 0.7314}{38}$$

$$\doteq 0.827$$

$$\therefore A \doteq 56°$$

$$\therefore C = 180° - 47° - 56°$$

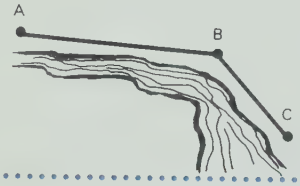
$$= 77°$$

If we used more accurate tables, we would find that A is somewhere between 55° and 56°.

Exercise 17.4

Written Solutions

1. Solve the following ABC triangles.
   (*a*) $a = 35, b = 27, \angle C = 72°$
   (*b*) $b = 5.3, c = 3.4, \angle A = 43°$
   (*c*) $a = 42, c = 37, \angle B = 112°$
   (*d*) $b = 38, c = 17, \angle A = 105°$
   (*e*) $a = 24, b = 18, c = 15$
   (*f*) $a = 7.2, b = 6.4, c = 8.5$
   (*g*) $a = 73, b = 56, c = 62$
   (*h*) $b = 123, c = 98, \angle A = 83°$
   (*i*) $a = 92, c = 78, \angle B = 48°$
   (*j*) $a = 18, b = 16, c = 22$



2. The goal posts in a regulation net are 183 cm apart. A player scores a goal from a point 10 m from one post and 11 m from the other. Within what angle does he make his shot?

3. A golf fairway is a "dog-leg" around the bend of a stream. If AB = 150 m and BC = 50 m and $\angle B = 143°$, how far would Bertha Birdiemaker have to hit the ball from A in order to reach the green? At what angle from AB would she have to hit her shot?

4. Town B is 12 km due east of town A and town C lies to the north of both A and B, 10 km from A and 8 km from B. At what angle from A does C lie? At what angle from B does C lie?

5. Two cars leave the same town at the same time. One goes north at 50 km/h and the other goes in a direction 47° east of north at the rate of 40 km/h. How far apart are they after 5 h?

6. Two sailboats leave a harbor at the same time. One sails in a direction 7° east of north at the rate of 15 km/h and the other in a direction 16° west of north at the rate of 20 km/h. What is the distance between them at the end of 2.5 h?

7. Linda Longstride and her pack of fearless Brownies arrive at point A. Their camp is at C, but to get there they have to choose between the long route through Cutshoe Mountains and the short route across Dismal Swamp. What a choice! How far is it from A to camp by each route?

1.(c) $b = 66, \angle A = 36°, \angle C = 32°$
(d) $a = 45, \angle B = 54°, \angle C = 21°$
(h) $a = 148, \angle B = 56°, \angle C = 41°$
(i) $b = 70, \angle A = 76°, \angle C = 56°$

2. 8°

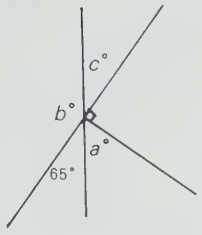
4. 41°, 56°

6. 21 km

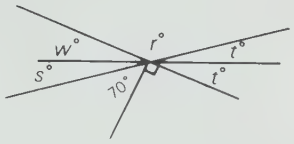
Maintaining Basic Skills 4

1. Find the value of the variables in each diagram in Question 1 and 2.

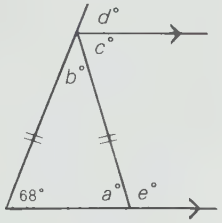
(*a*)



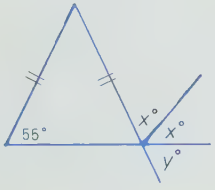
(*c*)



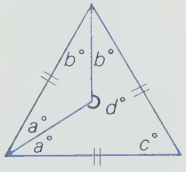
(*e*)



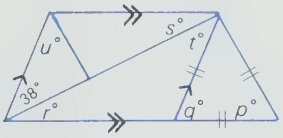
(*b*)



(*d*)

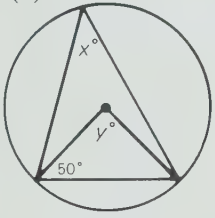


(*f*)

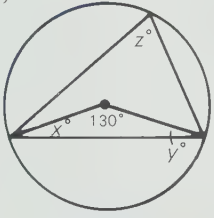


2.

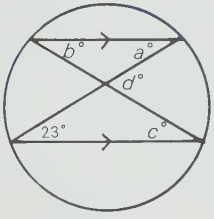
(*a*)



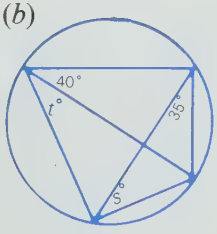
(*c*)



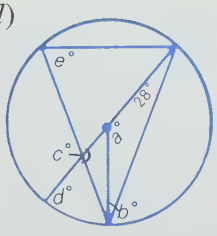
(*e*)



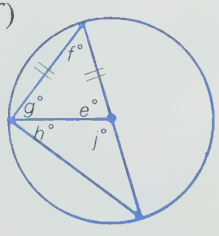
(*b*)



(*d*)



(*f*)



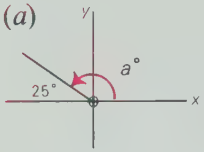
3. (*a*) Draw an equilateral triangle. What is the size of each angle?
   (*b*) Bisect each of the angles in the triangle.
   (*c*) Draw the inscribed circle of the triangle.

4. Without using a protractor, construct these angles.
   (*a*) 60° (*b*) 30° (*c*) 90° (*d*) 45°

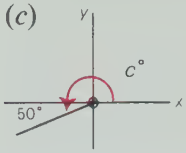
1(a) $a^\circ = 25^\circ$, $b^\circ = 115^\circ$, $c^\circ = 65^\circ$
(b) $x^\circ = 62\frac{1}{2}^\circ$, $y^\circ = 55^\circ$
(c) $r^\circ = 160^\circ$, $s^\circ = 10^\circ$, $t^\circ = 10^\circ$, $w^\circ = 10^\circ$
2(b) $s^\circ = 40^\circ$, $t^\circ = 35^\circ$
(c) $x^\circ = 25^\circ$, $y^\circ = 25^\circ$, $z^\circ = 65^\circ$
(e) $a^\circ = 23^\circ$, $b^\circ = 23^\circ$, $c^\circ = 23^\circ$

5. Give the number of degrees in each angle named with a variable.

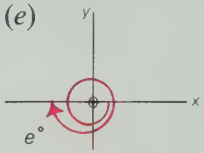
(a)



(c)



(e)



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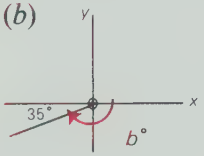
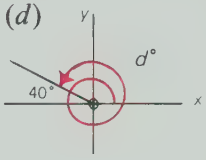
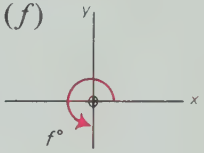


(d)

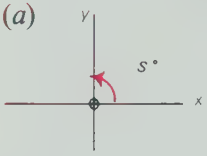


(f)

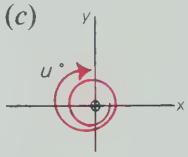


6. Give the number of radians in each angle marked.

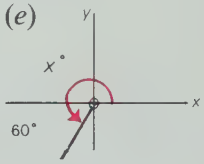
(a)



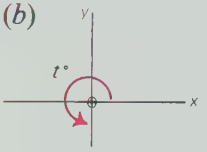
(c)



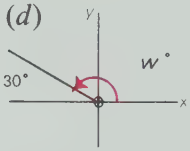
(e)



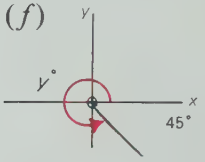
(b)



(d)



(f)



7. The following angles are in radians. Give the number of degrees in each.

(a) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (e) $\frac{11\pi}{12}$ (g) $-\frac{5}{2}\pi$

(b) $\frac{\pi}{4}$ (d) $-\frac{5\pi}{6}$ (f) 3π (h) $\frac{5\pi}{4}$

8. Express in radian measure in terms of π.

(a) 120° (c) 315° (e) −135° (g) 720°

(b) −30° (d) 240° (f) −390° (h) 150°

9. Sketch the following angles in standard position.

(a) 145° (c) 540° (e) $\frac{5}{4}\pi$ rad (g) 8π rad

(b) 340° (d) −100° (f) $-\frac{4}{3}\pi$ rad (h) $\frac{7}{4}\pi$ rad

10. Convert to degrees and minutes.

(a) 16.25° (b) 247.60° (c) 27.75° (d) 48.56°

11. Convert to degrees, minutes, and seconds.

(a) 37.47° (b) 186.36° (c) 19.321° (d) 95.854°

5.(a) $a° = 155°$
(b) $b° = -145°$
(c) $c° = 230°$

6.(a) $s° = \frac{\pi}{2}$ rad
(b) $t° = \frac{3}{2}\pi$ rad
(c) $u° = -\frac{7}{2}\pi$ rad

7.(a) 60°
(b) 45°
(c) 120°
(d) −150°

8.(a) $\frac{2}{3}\pi$ rad
(c) $\frac{7}{4}\pi$ rad
(g) 4π rad
(h) $\frac{5}{6}\pi$ rad

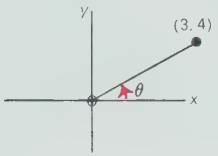
10.(a) 16° 15′
(b) 247° 36′

11.(c) 19° 19′ 15.6″
(d) 95° 51′ 14.4″

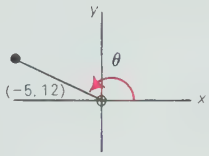
12. Convert to a decimal of a degree.
(*a*) 35°18′ (*b*) 152°50′ (*c*) 18°32′15″ (*d*) 39°17′45″

13. Give the values of the sine, cosine, and tangent of angle θ in each diagram.

(*a*)



(*c*)



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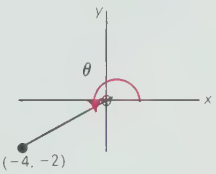
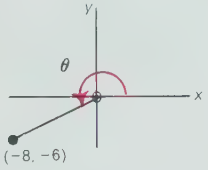
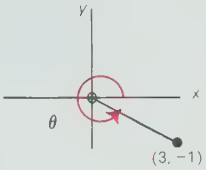


(*b*)



(*d*)



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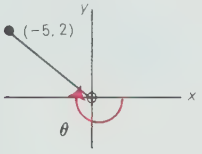


14. The terminal arm of an angle θ passes through the given point and θ is in standard position. Sketch a diagram and give the sine, cosine, and tangent of θ.
(*a*) (−4, 3) (*c*) (12, 5) (*e*) (8, −6)
(*b*) (2, −1) (*d*) (−3, −2) (*f*) (3, −5)

15. Draw a diagram of each angle in standard position. Mark suitable coordinates for a point on the terminal arm. Write the value of the three trig ratios of each angle.
(*a*) 0° (*d*) 60° (*g*) 210°
(*b*) 90° (*e*) 30° (*h*) 330°
(*c*) π rad. (*f*) 45° (*i*) −60°

16. Draw a diagram and find the other two trig ratios of θ.
(*a*) $\sin\theta = \frac{5}{13}$, θ is acute
(*b*) $\sin\theta = \frac{2}{3}$, θ in second quadrant (terminal arm of θ lies in second quadrant)
(*c*) $\cos\theta = -\frac{3}{5}$, θ in third quadrant
(*d*) $\cos\theta = \frac{\sqrt{3}}{2}$, θ in fourth quadrant
(*e*) $\tan\theta = -2$, θ in second quadrant
(*f*) $\sin\theta = \frac{1}{\sqrt{2}}$, θ in third quadrant

12(a) 35.3°
(b) 152.83°

13

	$\sin\theta$	$\cos\theta$	$\tan\theta$
(a)	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{4}{3}$
(c)	$\frac{12}{13}$	$-\frac{5}{13}$	$-\frac{12}{5}$

14(a) $\sin\theta = \frac{3}{5}$
$\cos\theta = -\frac{4}{5}$
$\tan\theta = -\frac{3}{4}$

(b) $\sin\theta = -\frac{1}{\sqrt{5}}$
$\cos\theta = -\frac{2}{\sqrt{5}}$
$\tan\theta = -\frac{1}{2}$

15(a) $\sin 0° = 0$
$\cos 0° = 1$
$\tan 0° = 0$

(d) $\sin 60° = \frac{\sqrt{3}}{2}$
$\cos 60° = \frac{1}{2}$
$\tan 60° = \sqrt{3}$

(g) $\sin 210° = -\frac{1}{2}$
$\cos 210° = -\frac{\sqrt{3}}{2}$
$\tan 210° = \frac{1}{\sqrt{3}}$

16(a)
$\cos\theta = \frac{12}{13}$
$\tan\theta = \frac{5}{12}$

(c)
$\sin\theta = \frac{-4}{5}$
$\tan\theta = \frac{4}{3}$

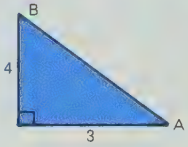
17. Sketch a diagram showing the two possible positions of the terminal arm. Find the other two primary trig ratios for each position of the angle.
   (*a*) $\sin A = \frac{3}{5}$ (*c*) $\tan C = \frac{2}{3}$ (*e*) $\tan E = -3$
   (*b*) $\cos B = -\frac{12}{13}$ (*d*) $\cos D = \frac{2}{5}$ (*f*) $\sin \theta = -\frac{1}{4}$

18. Find two values of each angle, for which
   (*a*) $\sin \theta = 0.643$ (*d*) $\sin \theta = -0.3215$
   (*b*) $\cos \theta = 0.3500$ (*e*) $\tan \theta = -0.8400$
   (*c*) $\tan \theta = 1.6200$ (*f*) $\cos \theta = -0.6314$

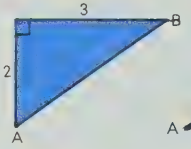
19. Express in terms of a positive acute angle.
   (*a*) sin 175° (*d*) sin 194° (*g*) cos 184°
   (*b*) cos (−25°) (*e*) tan 296° (*h*) sin 302°
   (*c*) tan 205° (*f*) cos 170° (*i*) tan (−260°)

20. State the three trig ratios of angles A and B.

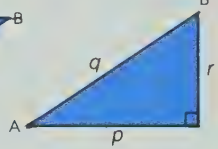
(*a*)



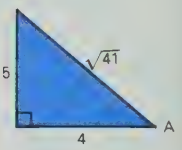
(*b*)



(*c*)

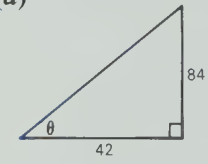


(*d*)

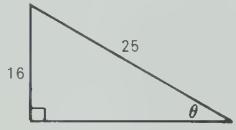


21. Solve for the angle marked.

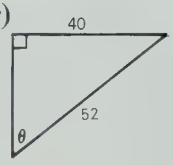
(*a*)



(*b*)



(*c*)



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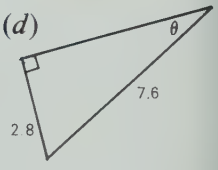
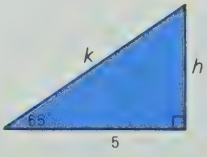


22. Find the lengths of the sides indicated by variables.

(*a*)



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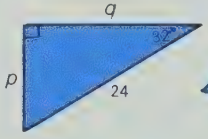

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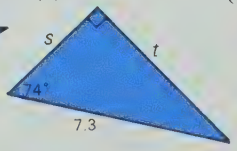

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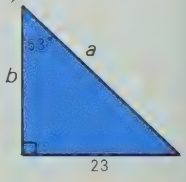


23. Solve the following triangles.
   (*a*) AB = 8.2, BC = 13.3, ∠B = 90°
   (*b*) PQ = 36, PR = 24, ∠R = 90°
   (*c*) ∠W = 90°, ∠T = 70°, ST = 15
   (*d*) ∠J = 90°, ∠K = 27°, JK = 6.3

18(d) 199°, 341°
(e) 140°, 320°
(f) 129°, 231°

19(a) sin 5°
(b) cos 25°
(c) tan 25°

20(a)
$\sin A = \frac{4}{5}$ $\sin B = \frac{3}{5}$
$\cos A = \frac{3}{5}$ $\cos B = \frac{4}{5}$
$\tan A = \frac{4}{3}$ $\tan B = \frac{3}{4}$

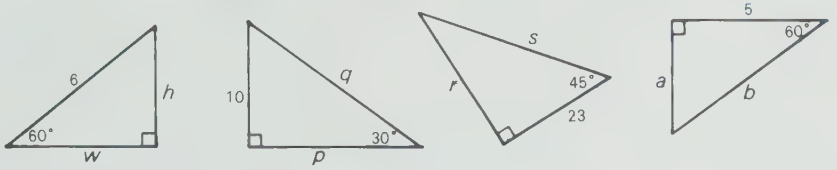
21(a) $\theta = 63°$
(b) $\theta = 40°$

22(c) $s = 2.01$
$t = 7.02$
(d) $a = 28.8$
$b = 17.3$

23(a) AC = 15.6, ∠A = 58°, ∠C = 32°
(b) QR = 26.8, ∠P = 48°, ∠Q = 42°

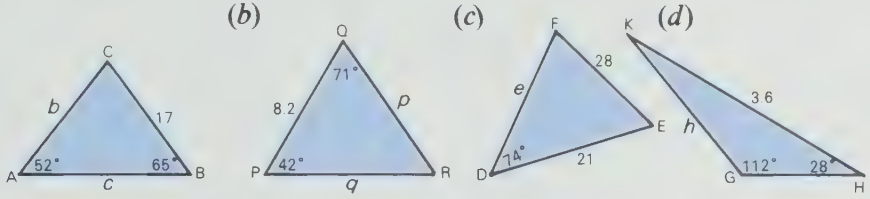
24. Without using tables, solve these triangles.

(*a*) (*b*) (*c*) (*d*)



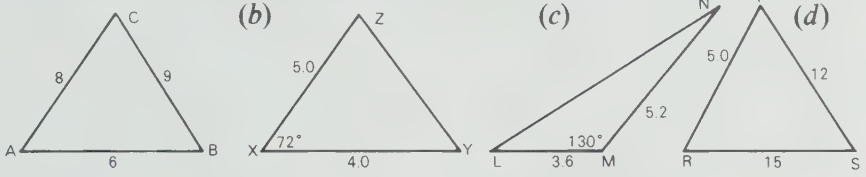
25. Solve the following triangles.

(*a*) (*b*) (*c*) (*d*)



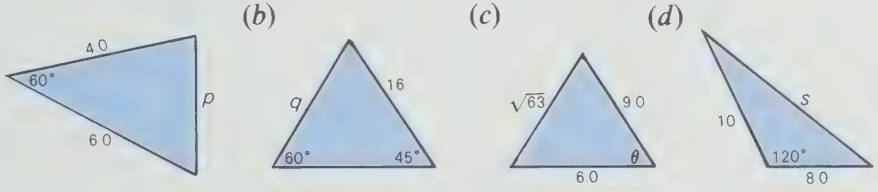
26. Solve the following triangles.

(*a*) (*b*) (*c*) (*d*)



27. Without using tables, solve for the variable indicated.

(*a*) (*b*) (*c*) (*d*)



28. Solve the following triangles, ABC.
   (*a*) $\angle A = 54°$, $\angle C = 73°$, $c = 19$
   (*b*) $\angle B = 49°$, $a = 4.1$, $c = 2.9$
   (*c*) $\angle B = 22°$, $\angle A = 80°$, $c = 23$
   (*d*) $a = 7.2$, $b = 9.4$, $c = 10.3$
   (*e*) $a = 0.81$, $b = 0.62$, $c = 1.23$
   (*f*) $\angle B = 55°$, $a = 27$, $b = 35$

29. Fred Flagsnapper measures a distance of 35 m from the foot of a flagpole. From this point he measures the angle of elevation, from his eye-level, to be 57°. His eye is 168 cm above the ground. How high is the flagpole?

30. From the top of a firetower 70 m high, spotter S. Beare measures the angle of depression of a forest fire to be 5°. How far away is the fire?

24(a) $w = 3$, $h = 3\sqrt{3}$, $\angle W = 30°$
(b) $p = 10\sqrt{3}$, $q = 20$, $\angle P = 60°$

25(c) $e = 25.2$, $\angle E = 60°$, $\angle F = 46°$
(d) $h = 1.8$, $k = 2.5$, $\angle K = 40°$

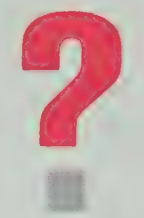
26(a) $\angle A = 78°$, $\angle B = 61°$, $\angle C = 41°$
(b) $YZ = 5.4$, $\angle Y = 63°$, $\angle Z = 45°$

27(a) $p = 2\sqrt{7}$
(c) $\theta = 60°$

28(a) $a = 16.1$, $b = 15.9$, $\angle B = 53°$
(b) $b = 3.1$, $\angle A = 86°$, $\angle C = 45°$
(c) $a = 23.2$, $b = 8.8$, $\angle C = 78°$

29. 55.6 m

Summary of Units 9 to 17



	Do you understand . . . ?	*Can you . . . ?*
Unit 9		
Lines and Their Equations		
9.1 Get-Set Exercise		
9.2 Slopes	the measure of steepness	calculate the rise and run of a line segment
	the meaning of rise and run	
	how to calculate slope	distinguish between a positive and negative slope
9.3 Equations of a Straight Line	how to tell when an equation represents a straight line	graph a line
	how to find the equation of a line if a point is on a line	find the equation of a line, given: (*a*) slope, point (*b*) two points
9.4 Equations of a Line in the Form $y = mx + b$	how to write equations in the form $y = mx + b$	find the equation of a line given the slope and the *y*-intercept
	what the "m" and "b" represent	
	the meaning of *y*-intercept	
Unit 10		
Linear Equations and Their Applications		
10.1 Direct Variation	the meaning of *y varies directly as x*	draw the graph of a direct variation
	how to establish the equation of a direct variation	
10.2 Partial Variation	the meaning of partial variation	write the equation of a partial variation and graph it
10.3 Inverse Variation	the meaning of *y varies inversely as x*	write the equation of an inverse variation and graph it

	Do you understand . . . ?	*Can you . . . ?*
Unit 11		
Coordinate Geometry		
11.1 Lengths of Line Segments Parallel to the Axes	when a line is parallel to the x axis when a line is parallel to the y axis	find the lengths of line segments parallel to: (a) the x axis (b) the y axis
11.2 Lengths of Line Segments Joining Any Two Points	the Pythagorean Theorem	apply the theorem to lengths of line segments write an equation to calculate the length of any line segment
11.3 Midpoints of a Line Segment	the meaning of midpoint	calculate the midpoint of a line segment (a) parallel to the axes (b) located anywhere on the plane state the general equation for finding the coordinates of the midpoint
Unit 12		
Second Degree Relations		
12.1 The Circle as the Graph of a Relation	the characteristics of a circle the meaning of domain and range	find the equation of a circle with centre the origin find the domain and range of a circle from the equation of a circle, state its centre and radius
12.2 Equation of Circles, Centre Not at the Origin	the difference between the equation of circles with centre at the origin, and centre not at the origin	find the equations of a circle with centre at (h, k) and radius r units
12.3 Inequations and Circles	the inequality signs the meaning of an inequation	graph an inequality involving a circle describe the graphs of inequalities with inequations
12.4 The Quadratic Relation	the meaning of a second-degree relation quadratic graphs of parabolas	graph a second-degree relation find the vertex and axis of symmetry of a parabola find the domain and range of a parabola

	Do you understand . . . ?	*Can you . . . ?*
Unit 13		
Angles		
13.1 Angle Review	how an angle is named	
	what is meant by vertex of an angle	
	adjacent, supplementary, complementary, and opposite angles	locate opposite angles
		name complementary and supplementary angles
13.2 Angles in a Triangle	the meaning of isosceles triangle, equilateral triangle	identify an isosceles and equilateral triangle
	the sum of the interior angles of a triangle	find the values of exterior angles of a triangle
	the meaning of exterior angle, interior opposite angle	find the values of interior angles of a triangle
13.3 Circles and Triangles	how to bisect an angle	draw the inscribed circle in a triangle
	how to draw a perpendicular bisector	draw the circumscribed circle of a triangle
13.4 Angles and Parallel Lines	the meaning of parallel lines and transversal	given parallel lines and a transversal, name corresponding angles that are equal and alternate angles that are equal
	the meaning of corresponding angles and alternate angles	
13.5 Angles in a Circle	when angles are in the same segment of a circle	name equal angles in a circle
13.6 Angles in a Polygon	the relationship between the number of sides of a polygon and the number of triangles that can be formed	find the sum of the interior angles of a polygon

	Do you understand . . . ?	*Can you . . . ?*
Unit 14		
Measures of Angles		
14.1 Measurement of Rotation	how many degrees there are in a revolution	use a protractor
	how many degrees there are in a right angle	
14.2 Units Smaller Than Degrees	how to change fractions of degrees to minutes and seconds	find the number of minutes in a degree
		find the number of seconds in a minute
14.3 Angles in Standard Position	the mean of an angle in standard position	
	coterminal angles	find coterminal angles
		distinguish between a positive and negative angle
14.4 The Radian Measure of an Angle	the meaning of a radian	
	the relationship among radians, arc length, and radius	change from radians to degrees or from degrees to radians
Unit 15		
Trigonometry		
15.1 Trigonometric Values	how to determine the trig values for any right triangle	write the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of x, y and r
		name the opposite side and adjacent side of an angle in a triangle
15.2 Trig Values of Special Angles	how to find the trig values for 30°, 45°, and 60° angles	evaluate expressions involving trig values for angles of 30°, 45°, and 60°
15.3 The Use of Trig Tables	how to use the trig tables	evaluate $\sin\theta$, $\cos\theta$, and $\tan\theta$ for any acute angle
		find the angle θ, given the value of $\sin\theta$, $\cos\theta$, or $\tan\theta$
15.4 Solution of Right Triangles	the meaning of solving a right triangle	find the values of the lengths of unknown sides and the measures of unknown angles of a triangle
15.5 Problems Involving Right-Angled Triangles	the meaning of angle of elevation and angle of depression	sketch a diagram of a word problem that involves angles

	Do you understand . . . ?	*Can you . . . ?*
Unit 16 **Trigonometric Values of Any Angle**		
16.1 Get Set	the sign convention for angles coterminal angles	determine the quadrant in which an angle lies from a point on its terminal arm
16.2 Trig Values for Any Angle	how the signs of the coordinates of a point affect the sign of the trig values	determine the other trig values, given one trig value of an angle and the quadrant in which the terminal arm lies
16.3 Signs of Trigonometric Ratios	the signs of the trig values in the four quadrants	apply the CAST rule
16.4 Trig Values of Angles Greater Than 90°	how to find the trig values of any angle by relating it to an acute angle	find the related acute angle for any angle find the trig value of the acute angle use the CAST rule to determine the sign
16.5 Trig Values of Any Angle Using Tables	how to find the related acute angle and thus determine the trig value	determine the correct sign for any related angle
Unit 17 **Graphing and Solution of Oblique Triangles**		
17.1 Trig Values of 0°, 90°, 180°, 270°, and 360°	how to find the trig values of angles whose terminal arm lies on one of the axes	evaluate all the trig values of 0°, 90°, 180°, 270°, and 360°
17.2 The Graphs of Sin θ, Cos θ, and Tan θ	what happens to the values of sin θ, cos θ, and tan θ as θ increases from 0° to 360°	sketch the graphs of sin θ, cos θ, and tan θ
17.3 The Law of Sines	the Laws of Sines and how it is used to solve triangles why there are two possible solutions in some cases	state the Law of Sines for $\triangle$ ABC interpret the problem by means of a sketch
17.4 The Law of Cosines	how to solve triangles using the Cosine Law	state the Law of Cosines

Review Exercise, Units 9 to 17

1. Find the equation of the line passing through the following points.
   (*a*) (3, 7), having a slope of 4
   (*b*) (1, −5), with a slope = $\frac{2}{3}$
   (*c*) (4, 8), parallel to the x axis
   (*d*) (2, −7), parallel to the y axis

2. Given P(2, 5), Q(4, 11). Find
   (*a*) the slope of PQ,
   (*b*) the equation of PQ,
   (*c*) the length of PQ.

3. Given R(4, 1), S(6, 9). Find
   (*a*) the equation of RS,
   (*b*) the midpoint of RS,
   (*c*) the length of RS.

4. Given A(−3, 1), B(1, 3). Find
   (*a*) the slope of AB,
   (*b*) the length of AB,
   (*c*) the midpoint of AB.

5. $3x - 6y = 12$. Find
   (*a*) the slope of the line,
   (*b*) the x-intercept,
   (*c*) the y-intercept.
   (*d*) Graph the line.

6. Repeat Question 5 for these lines.
   (*a*) $x + y = 6$
   (*b*) $2x + y = 8$
   (*c*) $4x - 3y = 6$

7. $2x + 7y = 28$. Find the following.
   (*a*) The coordinates of the points where the line crosses the x and y axes.
   (*b*) The equation of the line through (3, −1) having the same slope as the given equation.
   (*c*) The equation of the line having a slope of 3 and having the same x-intercept as the given equation.

8. $x - 3y = 12$. Find the following.
   (*a*) The x- and y-intercepts.
   (*b*) The equation of the line passing through (4, 2) and crossing the x axis at the same point as the given equation.
   (*c*) The equation of the line having the same y-intercept as the given equation, with a slope of $\frac{3}{4}$.

1(a) $4x - y - 5 = 0$
(b) $2x - 3y - 17 = 0$

2(a) 3
(b) $3x - y - 1 = 0$

3(b) (5, 5)

4(a) $\frac{1}{2}$
(b) $2\sqrt{5}$

5(a) $\frac{1}{2}$
(b) x-intercept = 4

6(a) slope = −1
x-intercept = 6
y-intercept = 6

7(a) (14, 0), (0, 4)
(b) $2x + 7y + 1 = 0$

8(c) $3x - 4y - 16 = 0$

9. $5x + 3y = 30$. Find the following.
   (*a*) The slope of the line.
   (*b*) The y coordinate of a point whose x coordinate is 3.
   (*c*) The equation of the line through the x-intercept parallel to the y axis.

10. The mass M of a pile of tiles on a wheelbarrow varies as the thickness T of the pile. A pile 15 cm high has a mass of 5 kg.
   (*a*) Find the equation relating M and T.
   (*b*) Graph the relation. What is the slope of the line?
   (*c*) What would be the mass of a pile 6 cm high?
   (*d*) What would be the height or thickness of a pile of tiles that had a mass of 8 kg?

11. The cost C of printing a school magazine is partly constant and partly varies as the number N of copies ordered. 500 copies cost \$1300. 700 copies cost \$1700.
   (*a*) Find the equation.
   (*b*) Graph the relation, using N as the horizontal axis.
   (*c*) What is the value of the C intercept?
   (*d*) What would be the cost of 400 copies?
   (*e*) How many copies could be ordered for \$700?

12. How many radians are equivalent to
   (*a*) 90° (*b*) 75° (*c*) 540° (*d*) −135°

13. How many degrees are equivalent to
   (*a*) 2π rad (*b*) $\frac{7}{6}\pi$ rad (*c*) $-\frac{5}{3}\pi$ rad (*d*) 5π rad

14. The terminal arm of an angle θ passes through $(-6, 8)$. Find $\sin\theta$, $\cos\theta$, $\tan\theta$.

15. $\sin\theta = -\frac{2}{3}$. Draw a diagram showing the two possible positions for the terminal arm of θ. For each position give the value of $\cos\theta$, $\tan\theta$.

16. Express in terms of a positive acute angle.
   (*a*) cos 250° (*b*) sin(−17°) (*c*) tan 117° (*d*) cos 145°

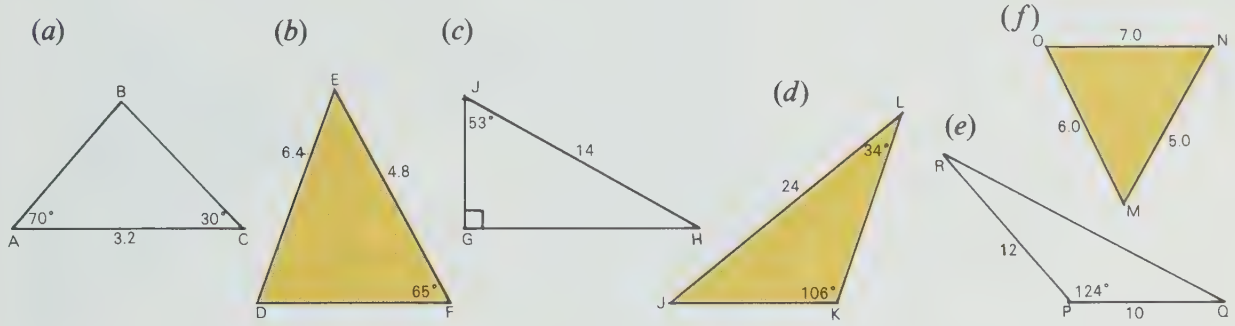
9(b) 5
(c) $x = 6$
10(a) $M = \frac{1}{3}T$
(b) slope $= \frac{1}{3}$
(c) 2 kg
(d) 24 cm
11(c) 300
(d) \$1100
12(a) $\frac{\pi}{2}$ rad
(b) $\frac{5\pi}{12}$ rad
13(a) 360°
(b) 210°
14. $\sin\theta = \frac{4}{5}$
15. $\cos\theta = \frac{\sqrt{5}}{3}$
$\cos\theta = -\frac{\sqrt{5}}{3}$
16(a) −cos 70°
(b) −sin 17°

17. Draw a diagram for each angle. Mark suitable coordinates on the terminal arm. Without using tables find the value of

(*a*) $\cos 30°$ (*c*) $\tan 135°$ (*e*) $\sin \frac{\pi}{2}$ (*g*) $\sin 0°$

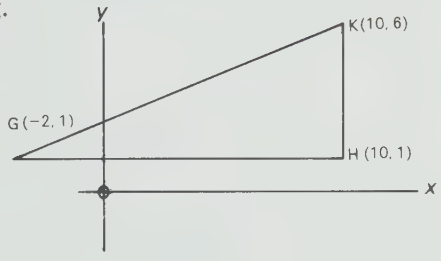
(*b*) $\sin 60°$ (*d*) $\sin 240°$ (*f*) $\cos \pi$ (*h*) $\tan 270°$

18. Solve the following triangles.

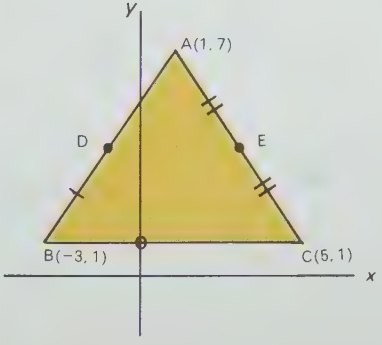


19. From the diagram, find the following.

(*a*) the equation of GH
(*b*) the equation of HK
(*c*) the length of GH, HK, GK
(*d*) the midpoint of GK
(*e*) the measure of the angle KGH



20. Find the following



(*a*) The equation of the line through A, parallel to BC.

(*b*) The length of AB, BC, CA. Is the triangle isosceles? Is it equilateral?

(*c*) The coordinates of D, the midpoint of AB; and of E, the midpoint of AC.

(*d*) The equation of DE, and of BC. Compare the slopes of these lines.

(*e*) The length of DE and of BC. Compare these lengths.

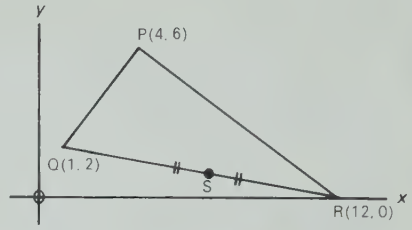
17(a) $\cos 30° = \frac{\sqrt{3}}{2}$
(b) $\sin 60° = \frac{\sqrt{3}}{2}$
(g) $\sin 0° = 0$
(h) $\tan 270°$ **is undefined**

18(b) **FD = 6.7, ∠E = 72°, ∠D = 43°**
(c) **∠H = 37°, JG = 8.4, GH = 11.2**
(d) **∠J = 40°, KL = 16, JK = 14**

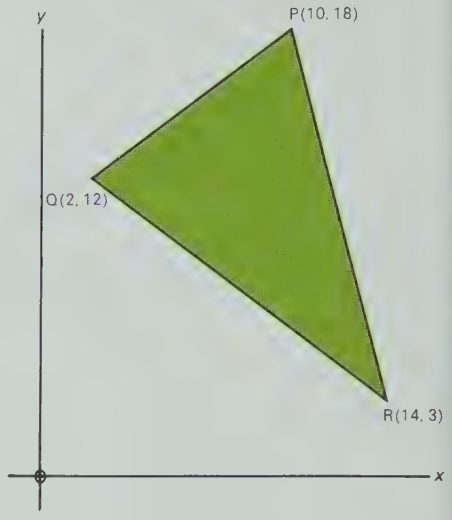
19(a) $y = 1$
(b) $x = 10$
(c) **12, 5, 13**

20(c) **D(−1, 4), E(3, 4)**
(d) **DE has equation $y = 4$. BC has equation $y = 1$. Slopes equal, each = 0.**

21. (*a*) Find the lengths of PQ, PR, QR. From these prove that $\angle P = 90°$.
(*b*) Find the coordinates of S, where QS = SR.
(*c*) Find the lengths of QS, SR, SP. What do you observe?
(*d*) Find the equation of a line through S having the same slope as PQ.



22. (*a*) Find the length of each side of the triangle PQR.
(*b*) Find the measure of $\angle PQR$.

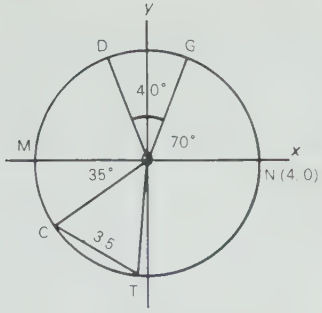


23. Mark and Harry are 120 m apart on a beach. They want to know how far away they are from an island in the lake. The angle Mark-Harry-Island is measured to be 58°, and the angle Harry-Mark-Island is 75°. What is the distance from Mark to the island?

24. An engineer has to cut a road through dense bush. Two existing roads intersect at an angle of 65°. The new road runs from a point 6.2 km from the intersection on one of the roads to a point 8.5 km from the intersection on the other road. How long will the new road be?

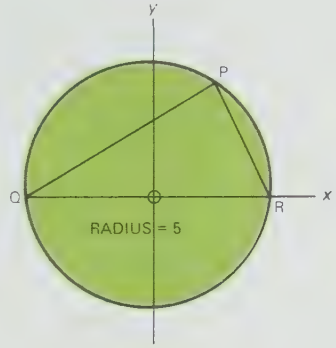
25. Towns A and B are 8.0 km apart, B and C are 5.0 km apart, and A and C are 6.4 km apart. George lives at town B. He has his TV antenna pointed toward A. Through what angle must he rotate his antenna to pick up the TV station at C?

21(a) $PQ = 5$, $PR = 10$, $QR = 5\sqrt{5}$
(d) $4x - 3y - 23 = 0$
22(a) $PQ = 10$, $QR = 15$, $PR = \sqrt{241} \doteq 15.52$
23. 139 m
25. 53°

26. 

Find
(*a*) the equation of the circle
(*b*) the length of DG
(*c*) ∠DOM, ∠ODG
(*d*) ∠COT
(*e*) ∠TON

27. (*a*) Find the equation of the circle.
(*b*) Find y, where P$(4, y)$ lies on the circle.
(*c*) Find the lengths of QP, PR, QR. Show one angle in $\triangle PQR = 90°$.
(*d*) Find ∠PQR.
(*e*) What is the equation of QP?



28. (*a*) Quadrilateral ABCD has vertices A$(2, 8)$, B$(6, 2)$, C$(14, 6)$, D$(8, 12)$. Sketch ABCD.
(*b*) Find the coordinates of P, the midpoint of AB; of Q, the midpoint of BC; of R, the midpoint of CD; and of S, the midpoint of DA.
(*c*) Find the length of PS and of QR. Compare these lengths.
(*d*) Find the slope of PS and of QR, and compare them.
(*e*) Compare the lengths of PQ and SR.
(*f*) Compare the slopes of PQ and SR.

29. For the end of a house ABCDE, find the following.
(*a*) slope of the roof AE, and of ED
(*b*) equation of the roof ED, of the wall DC, and of AD
(*c*) length of a brace AC
(*d*) length of the brace FG
(*e*) angle EDA and angle EAD
(*f*) Show that EAD is an isosceles triangle.

30. A bicycle wheel is described by $x^2 + y^2 = 144$.
(*a*) What is the radius of the wheel?
(*b*) The bicycle is ridden through a mud puddle. Some mud sticks to the wheel for $\frac{1}{6}$ of a revolution and then drops off. Through how many degrees and how many radians did the mud rotate?

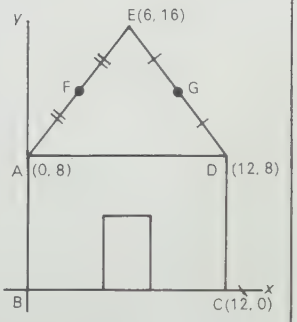
26(a) $x^2 + y^2 = 16$
(d) 52°

27(a) $x^2 + y^2 = 25$
(b) $y = 3$

28(b) P(4, 5)
Q(10, 4)
R(11, 9)
S(5, 10)
(d) Slope of PS = 5.
Slope of QR = 5.

29(a) Slope of AE = $\frac{4}{3}$,
Slope of ED = $-\frac{4}{3}$.
(b) Equation of ED is $4x + 3y = 72$, of DC is $x = 12$, of AD is $y = 8$.
(e) ∠EDA = 53°,
∠EAD = 53°

30(a) 12 units
(c) $25\frac{1}{7}$ units



(*c*) Another piece of mud remained on during a rotation of 120°. Through what distance did it travel in the circle?

(*d*) How far was the piece of mud from the ground in (*c*) when it dropped off?

31. (*a*) A lake is described by $x^2 + y^2 < 100 \cap x > 0 \cap y > 0$. Draw a diagram of the lake.

(*b*) Bruce Burchbark has a cottage at (4, 0), and Tim Timberlake has one at (0, 8). Mark them on the diagram. How far apart are the cottages?

(*c*) What is the equation of the line joining the cottages?

(*d*) What acute angle does the line in (*c*) make with the x shore line?

(*e*) There is an island at (6, 5) that Bruce sails to in his boat. What is the equation of his path?

(*f*) How far did Bruce travel to get to the island?

(*g*) From the island Bruce sails in a line having the same slope as the line joining the cottages. What is the equation of his path?

(*h*) Where does Bruce reach the x axis shore line?

(*i*) Tim leaves his cottage and water skis parallel to the x axis shore. What is his equation? Where would he reach the opposite shore?

(*j*) Halfway between his cottage and the opposite shore, Tim fell into the water and had to be picked up by the towing boat. How far did he ski? At what point did he fall in?

32. The volume of an ice cream cone varies as the square of its radius, when its height remains unchanged. A cone with radius 3 cm has a volume of 108 cm^3. Find the volume of an ice cream cone having radius 3.5 cm.

33. The mass of a carton of soap flakes varies as the square of one edge of its base, when its height does not change. One carton has a base edge of 1 m and a mass of 8 kg.

(*a*) A second carton has a base edge of 150 cm. What is its mass?

(*b*) A third carton has a mass of 30 kg. What is the length of its base edge?

31(c) $2x + y - 8 = 0$
(d) 63°
(g) $2x + y - 17 = 0$

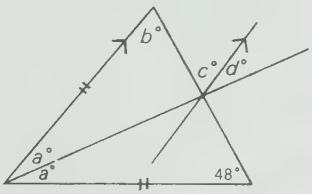
32. 147 cm^3

34. The number of chocolate bars you can buy with your allowance varies inversely as the price of each. If you can buy 150 bars at 10¢ each with your allowance, how many would you get if each was 12¢?

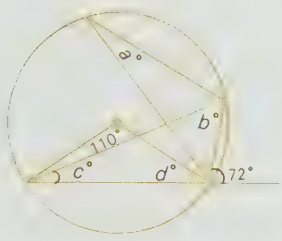
35. Robin is holding the string attached to his bat kite. The string is 100.0 m long. The angle between the string and the horizontal is 55°. Find the height of the kite above his hand.

36. A bird is sitting on top of a TV tower 30.0 m high. At a point 20.0 m from the tower Lesley Lenscap, standing on the ground, measures the angle of elevation of the bird. What should her measurement be? She is going to take a picture of the bird, so she needs to know how far away it is. What distance does she calculate?

37. Find the value, in degrees, of each variable.



38. Find the measure of each angle marked with a variable.



39. A new subdivision, Cartesian Slopes, is laid out in the xy plane. Mark these streets using the units given.
   (*a*) First Street is $y = 0$. Sixth Avenue is $x = 0$.
   Third Street is $y = 2$. Fourth Avenue is $x = 4$.
   (*b*) Apple Street is $y = x^2 + 2$. Grove Street is $y = \frac{1}{4}x^2$.
   Pear Street is $x = -y^2$.
   (*c*) Circle Street is $(x + 3)^2 + (y - 2)^2 = 4$.
   Round Street is $x^2 + y^2 = 16$.
   (*d*) Russell Street is a circle of radius 1 and centre $(4, -2)$.
   What is its equation?

34. 125

36. 56°, 36 m

38. $a = 17$, $b = 55$, $c = 17$, $d = 35$

39(d) $(x - 4)^2 + (y + 2)^2 = 1$

Interest . . . . . $

Principal . . . . . $

Interest rate . . . %

Time . . . . . years

Unit 18

Simple and Compound Interest

18.1 Get-Set Exercise

Per cents are special rates that indicate the ratio of some number to 100. Thus, 20% can be expressed as $\frac{20}{100} = 0.20$. Therefore, we can express per cent either as a fraction or as a decimal.

1. Express the following per cents in fractional form.
   (*a*) 13% (*d*) $\frac{1}{4}$% (*g*) $\frac{1}{2}$%
   (*b*) 0.5% (*e*) 0.05% (*h*) $3\frac{1}{4}$%
   (*c*) 50% (*f*) 1% (*i*) $33\frac{1}{3}$%

2. State the following ratios as per cents.
   (*a*) $\frac{1}{4}$ (*d*) $\frac{3}{8}$ (*g*) $\frac{2}{3}$
   (*b*) $\frac{3}{4}$ (*e*) 12:16 (*h*) $\frac{1}{10}$
   (*c*) $\frac{1}{2}$ (*f*) 5:15 (*i*) 1:20

3. Express as per cents.
   (*a*) 0.4 (*d*) 0.50 (*g*) 0.90
   (*b*) 0.18 (*e*) 0.25 (*h*) 0.75
   (*c*) 0.46 (*f*) 0.375 (*i*) 1.33

4. Here are the marks Gina got on her term tests. Find her per cent in each subject.
   History . . . . . 36 out of 40
   Geography . . . . 18 out of 25
   Biology . . . . . 37 out of 50
   English . . . . . 45 out of 75
   Mathematics . . . 21 out of 30

18.2 Simple Interest

When money is borrowed there is a charge for the use of the money until it is repaid. This charge is called the interest on the money borrowed and is usually denoted by I. The sum borrowed is called the principal (P). The length of time the money is borrowed is called the time of the loan (T), and the interest rate (R) is the percentage of the sum borrowed that must be paid each year for the use of the money. It is always understood to be interest per year (per annum) unless stated otherwise.

Example 1

What is the interest charged on a loan of \$100 borrowed for 1 year at 10% interest?

Solution

$$\begin{aligned} \text{Interest} &= \text{Principal} \times \text{Interest Rate} \times \text{Time} \\ I &= P \times R \times T \\ &= \$100 \times \frac{10}{100} \times 1 \\ &= \$10 \end{aligned}$$

$\therefore$ interest charged for 1 year is \$10.

Example 2

Penelope Pennypincher lends a sum of \$1000 on March 12 that is repaid on October 17 of the same year. Find the interest that Penny must be paid if the interest rate is 8%.

Solution

From March 12 to October 17:

Mar. 12 to 31	= 19 days
April	= 30 days
May	= 31 days
June	= 30 days
July	= 31 days
Aug.	= 31 days
Sept.	= 30 days
Oct. 1 to 17	= 17 days
Total days	= 219, $\therefore T = \frac{219}{365}$

$$\text{Now } I = P \times R \times T$$
$$= \$1000 \times \frac{8}{100} \times \frac{219}{365}$$
$$= 80 \times \frac{3}{5}$$
$$= \$48$$
$\therefore$ interest charge is \$48.

The amount of money that will repay the loan is therefore the sum of the money borrowed (P) plus the interest (I). If A denotes this amount to be repaid, we can say $A = P + I$.

If $A = P + I$,
then $A = P + PRT$
$= P(1 + RT)$

Example 3

Find the amount of money required to repay a loan of \$750. The money was borrowed at 8% interest rate for a time period of 73 days.

Solution

$$I = P \times R \times T$$
$$= \$750 \times \frac{8}{100} \times \frac{73}{365}$$
$$= 7.50 \times 8 \times \frac{1}{5}$$
$$= 1.50 \times 8$$
$$= \$12.00$$
$$\therefore A = P + I$$
$$= \$750 + \$12$$
$$= \$762$$
$\therefore$ the amount of money required to repay the loan would be \$762.

Why $\frac{73}{365}$?

Why 2 decimal places?

Exercise 18.2

Written Solutions

1. State the interest for 1 year.
   (*a*) \$100 at 5%
   (*b*) \$2000 at 10%
   (*c*) \$250 at 8%
   (*d*) \$630 at 7%
   (*e*) \$125 at 9%
   (*f*) \$500 at 18%

1.(a) \$5.00
(b) \$200.00
(c) \$20.00

2. Calculate the interest in each of the following.

	Principal	*Rate*	*Time*
(*a*)	\$6000	9%	3 months
(*b*)	\$430	$8\frac{1}{2}$%	1 year
(*c*)	\$125.40	10%	1 month
(*d*)	\$2500	18%	5 weeks
(*e*)	\$514.80	15%	73 days
(*f*)	\$83.40	24%	140 days
(*g*)	\$50.00	20%	200 days
(*h*)	\$562.17	15%	300 days
(*i*)	\$1000	9%	1 year
(*j*)	\$270	$5\frac{1}{2}$%	2 years

3. Calculate the amount that must be repaid in each of the following loans.

	Principal	*Rate*	*Time*
(*a*)	\$730	6%	73 days
(*b*)	\$1380	$9\frac{3}{4}$%	12 years
(*c*)	\$480	10%	146 days
(*d*)	\$875	5%	4 years 11 months
(*e*)	\$2190	$8\frac{3}{4}$%	219 days
(*f*)	\$1200	9%	2 months
(*g*)	\$1460	$7\frac{3}{4}$%	219 days
(*h*)	\$1000	8%	1 month
(*i*)	\$660	$7\frac{1}{4}$%	1 year 8 months
(*j*)	\$2500	8%	10 days

4. Miss Ima Indebt borrows \$2400 from her bank for 2 years at 9% interest. Find the amount Miss Indebt must repay to the bank.

5. To buy a boat Peter Cruser borrows \$2190 from his bank on June 23. How much must he repay on September 4th if money is worth 9% per annum?

6. How much interest should be charged for borrowing \$350 for 2 years at 8% per annum?

7. Find the amount of money that can be collected after 5 years if \$6735 has been earning simple interest at the rate of $6\frac{1}{2}$% per annum.

8. Wayne Wantmor borrows \$1400 for 11 months from the Sharkee Financial Corp. at an annual rate of 24%. Determine the amount that Wayne must repay.

2.(f) \$7.68
(g) \$5.48
(h) \$69.31
(i) \$90.00
(j) \$29.70

3.(a) \$738.76
(b) \$2994.60
(c) \$499.20
(d) \$1090.10
(e) \$2304.98

4. \$2832.00

6. \$56.00

8. \$1708.00

18.3 Simple Interest Relationships

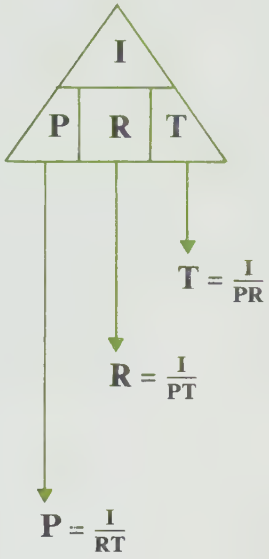
$I = P \times R \times T$ is a relation among the variables principal, rate of interest, the time in years, and the interest. In order to solve for one of the variables the values of the other three must be known.

(1) $I = P \times R \times T$ where we know the values of P, R, T.

(2) $P = \dfrac{I}{RT}$ where the values of I, R and T are known.

(3) $R = \dfrac{I}{PT}$ where the values of I, P and T are known.

(4) $T = \dfrac{I}{RP}$ where the values of I, R and P are known.



Example 1

A student, Prudence Pencilpusher, borrows \$200 from the bank and pays \$15 interest on the loan 1 year later. What interest rate was she charged?

Solution

$$I = PRT$$

where $P = 200$, $I = 15$, $T = 1$

$$\therefore R = \frac{I}{P \times T} = \frac{15}{200 \times 1} = \frac{7.5}{100}$$

$\therefore$ the rate of interest is 7.5%

Example 2

Pete Paperwate invested \$1000 at $7\frac{1}{2}\%$ interest. How long did it take for him to collect \$75 interest?

Solution

$$P = \$1000,\ I = \$75,\ R = 7\frac{1}{2}\%$$

$$\therefore T = \frac{75}{1000 \times \frac{7.5}{100}} = \frac{75}{10 \times 7.5} = 1$$

$\therefore$ it would take one year for him to earn \$75 interest.

$R = \frac{I}{?}$

Exercise 18.3

Written Solutions

1. Calculate the rate of interest as a percent in each of the following.

	Time	*Principal*	*Interest*
(*a*)	100 days	\$5000	\$342.53
(*b*)	4 months	\$2000	\$150.00
(*c*)	173 days	\$1240	\$14.88
(*d*)	292 days	\$1520	\$50.16
(*e*)	4 years	\$2785	\$668.40
(*f*)	$5\frac{1}{4}$ years	\$675	\$194.81

2. Calculate the principal in each of the following.

	Time	*Rate of Interest*	*Interest*
(*a*)	2 years	8%	\$20
(*b*)	73 days	15%	\$12.15
(*c*)	7 years	$6\frac{1}{2}$%	\$510
(*d*)	6 years 2 months	7%	\$1579
(*e*)	50 days	7%	\$14
(*f*)	3 years	$8\frac{1}{2}$%	\$1255

3. Calculate the time to the nearest day in each of the following.

	Rate of Interest	*Principal*	*Interest*
(*a*)	10%	\$1000	\$50
(*b*)	9%	\$542.16	\$34.17
(*c*)	8%	\$88.88	\$1.76
(*d*)	$9\frac{1}{2}$%	\$1000	\$95
(*e*)	18%	\$800	\$43.50
(*f*)	10%	\$462	\$4.62

4. The Doit Yourself Nursery borrows \$2500 from the bank and agrees to repay in 18 months. If the interest rate on the loan was $8\frac{1}{2}$%, what was the interest on the loan, and what amount would have to be repaid at the end of the 18 months?

5. What principal should be invested now to have \$40 interest in 6 months if the investment earns 8% per annum?

6. How long will it take \$150 deposited in a bank at 4% to earn \$1.50 in interest?

1.(d) $4\frac{1}{8}$%
(e) 6%
(f) 5.5%

2.(d) \$3657.92
(e) \$1460
(f) \$4921.57

3.(d) 365
(e) 110
(f) 37

4. Interest was \$318.75
Amount would be \$2818.75.

6. 91 days

7. (*a*) On the same pair of axes, draw two graphs showing interest (in dollars) against time (in days) for the simple interest on \$100 for a period of one year, using the following rates.
   (*i*) 5% (*ii*) 8%
   (*b*) Use the graphs of part (*a*) to answer the following questions.
   (*i*) What is the interest (to the nearest 10 cents) on \$100 at 5% for 60 days? 90 days? 250 days?
   (*ii*) What is the interest (to the nearest dollar) on \$500 at 8% for 110 days? 210 days? 310 days?
   (*iii*) What time (to the nearest 5 days) will be required for \$100 to earn \$4.60 interest at 5%? at 8%?

Hint $I = PRT$
$\therefore I = 100(R)T$

18.4 Compound Interest

When discussing compound interest businessmen frequently use the expression "interest earns interest." This is the essential difference between simple interest and compound interest. In simple interest calculations the principal remains constant, whereas in compound interest calculations the principal increases at the end of every interest period. In actual practice, simple interest is rarely used beyond one interest period. However, it is useful in dealing with interest based on fractions of an interest period.

A Suppose we deposit \$10 in a bank and the bank pays 4% interest compounded annually. What is the amount at the end of the first three years?

At the end of:		*Amount*
the 1st year	$A = 10 + 0.04(10)$ $= 10(1 + 0.04)$	\$10.40
the 2nd year	$A = 10.40 + 0.04(10.40)$ $= [10.40](1 + 0.04)$ $= [10(1 + 0.04)](1 + 0.04)$ $= 10(1.04)\ (1.04)$ $= 10(1.04)^2$	\$10.82
the 3rd year	$A = 10(1.04)^2 + 0.04[10(1.04)^2]$ $= 10(1.04)^2(1 + 0.04)$ $= 10(1.04)^2(1.04)$ $= 10(1.04)^3$	\$11.24
the nth year	$A = 10(1.04)^n$	

$A = P + I$
$= P + PRT$
$= P(1 + RT)$

Example 1

If \$100 is deposited in a bank that pays interest at 6% per year, what is the amount after the 1st year, 2nd year, 5th year, and 10th year?

Solution

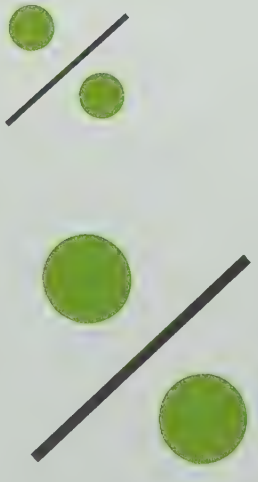
At the end of:			*Amount*
the 1st year	$100 + (0.06)(100)$		
	$= 100(1 + 0.06)$	$=$	$100(1.06)$
the 2nd year	$100(1.06) + 0.06[(100)(1.06)]$		
	$= (100)(1.06)(1 + 0.06)$		
	$= 100(1.06)(1.06)$	$=$	$100(1.06)^2$
the 5th year	$100(1.06)^4 + (0.06)[(100)(1.06)^4]$		
	$= 100(1.06)^4\ (1 + 0.06)$		
	$= 100(1.06)^4\ (1.06)$	$=$	$100(1.06)^5$
the 10th year	$100(1.06)^9 + (0.06)[(1.00)(1.06)^9]$		
	$= 100(1.06)^9(1 + 0.06)$		
	$= 100(1.06)^9(1.06)$	$=$	$100(1.06)^{10}$

Example 2

In general terms, if \$$P$ is allowed to accumulate interest at i% per year, what is the amount after the 1st year, 2nd year, 5th year, and the nth year?

Solution

At the end of:		*Amount*
the 1st year	$P + i\%$ of P	$= P(1 + i)$
the 2nd year	$P(1 + i) + i\%$ of $P(1 + i)$	
	$= P(1 + i)(1 + i)$	$= P(1 + i)^2$
the 5th year	$P(1 + i)^4 + i\%$ of $P(1 + i)^4$	
	$= P(1 + i)^4(1 + i)$	$= P(1 + i)^5$
the nth year	$P(1 + i)^{n-1} + i\%$ of $P(1 + i)^{n-1}$	
	$= P(1 + i)^{n-1}\ (1 + i)$	$= P(1 + i)^n$



This is called the general formula for compounding principals.

$$A = P(1 + i)^n$$

A is the amount of money you have accumulated after n interest periods.

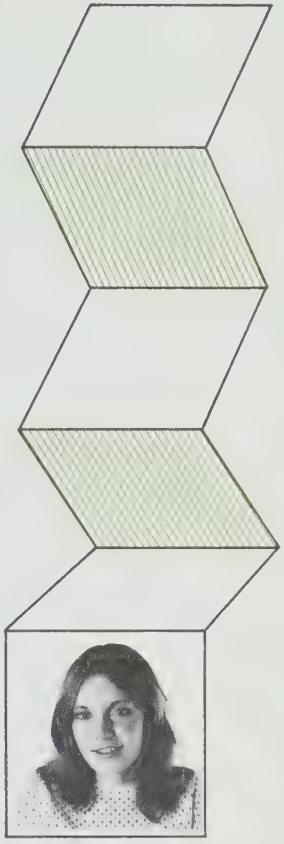
P is the sum of money you begin with (the principal).

i is the interest rate for each interest period, expressed as a decimal.

n is the number of interest periods.

1. Copy and complete the following table.

Principal	*Rate of Interest*	*Amount after 14 years*
\$10	5% compounded yearly	$10(1.05)^{14}$
\$8	8% compounded annually	
\$100	9% per annum	
\$250	7% per year	



B If the interest is compounded twice a year, then the interest rate for each interest period will be $\frac{1}{2}$ of the interest rate for the whole year. For example, if 5% represents the interest rate per year, the rate for 6 months will be $2\frac{1}{2}\%$, the rate for 3 months $1\frac{1}{4}\%$, and the rate for each month $\frac{5}{12}\%$.

Example

What is the interest rate per period if a sum of money is invested at the rate of 6% compounded (*a*) semi-annually? (*b*) every 4 months?

Solution

(*a*) Since interest is calculated twice in a year, the interest rate each interest period is $\frac{1}{2}$ of 6%.

$$i\% = \frac{6}{2} = 3\%, \ \therefore i = 0.03$$

(*b*) Since interest is calculated three times in a year, the interest rate each interest period will be $\frac{1}{3}$ of 6%.

$$i\% = \frac{6}{3} = 2\%, \ \therefore i = 0.02$$

1. State the information missing from the following table.

Interest per annum	*Compounded*	*Interest per period*
8%	semi-annually	
9%	monthly	
4%	every 3 months	
10%	every 4 months	
7%		$\frac{7}{12}\%$
12%		4%
2%		1%
	quarterly	2%
	every 4 months	3%
	monthly	2%

C Now let's look at different lengths of time combined with various interest rates.

Example 1

If \$100 is deposited for 3 years in a bank at 6% interest compounded semi-annually, calculate the amount of money in the bank at the end of 3 years.

Solution

$$A = P(1 + i)^n$$

where P is \$100

$$i \text{ is } \frac{6}{2} = 3\% \text{ per interest period}$$

$$n \text{ is } 3 \times 2 = 6 \text{ interest periods in 2 years}$$

$$\therefore A = 100(1 + 0.03)^6$$

$$= 100(1.03)^6$$

We will leave evaluation of expressions like this for the next section.

Suits me!

1. Copy and complete the following table.

Principal	*Compound Interest Rate*	*Time* (years)	*Number of Interest Periods*	*Interest per Period*	*Amount*
\$1	4% annually	2			$1(1.04)^2$
\$10	4% semi-annually	2			
\$100	4% quarterly	2			
\$8	6% monthly	2			

Therefore if we invest $\$P$ at interest rate r and the interest is compounded t times a year for x years, the amount (A) we will have at the end of this time can be calculated using the formula

$$A = P(1 + i)^n$$

where $i = \frac{r}{t}$ (interest per period)

and $n = xt$ (number of interest periods)

Exercise 18.4

Written Solutions

1. Copy and complete the following table to find the total amount after 10 years if \$100 is invested at 6% compounded semi-annually. Do not simplify your answers.

10 years = 20 interest periods.

Interest Calculation Period	*Principal at Start of Period*	*Amount at End of Period*
1	100	100(1.03)
2	100(1.03)	$100(1.03)^2$
⋮	⋮	⋮
19		
20		

2. Copy and complete the following table.

Annual interest rate (i)	6%	8%	10%	12%	9%	4%	7%	5%	3%
Number of times the interest is calculated per year	4	2		12		4	3	12	
Interest rate for each period	1.5%		2.5%		3%				1%

Do not simplify your answers.

3. Find an expression for the amount in each of the following.
   (*a*) \$2000 which collects interest at 5% per annum, compounded annually for 10 years
   (*b*) \$300 which collects interest at 8% per annum, compounded quarterly for 4 years
   (*c*) \$25 at 7%, compounded annually for 20 years
   (*d*) \$250 at 12%, compounded monthly for 3 years
   (*e*) \$1500 at 8%, compounded quarterly for $3\frac{1}{2}$ years
   (*f*) \$110 at 4%, compounded semi-annually for 15 years
   (*g*) \$5000 at $1\frac{1}{2}$%, compounded semi-annually for $5\frac{1}{2}$ years
   (*h*) \$28 at 6%, compounded monthly for 6 years
   (*i*) \$750 at 9%, compounded quarterly for 10 years
   (*j*) \$900 at 10%, compounded semi-annually for $4\frac{1}{2}$ years

3.(a) $2000 (1.05)^{10}$
(b) $300 (1.02)^{16}$
(c) $25 (1.07)^{20}$
(d) $250 (1.01)^{36}$
(e) $1500 (1.02)^{14}$

$(1.04)^{10}$ looks tedious, all right!

18.5 Compound Interest Tables

For convenience and in order to eliminate tedious calculations for quantities such as $(1.04)^{10}$, tables have been provided on pages 466-467. These tables give the amount (A) that a principal (P) of \$1 will compound to at certain rates of interest. A portion of the amount table is shown.

Amount for 1
$(1 + i)^n$

n	4%	5%	6%	7%	8%	9%	10%	n
1	1.04000	1.05000	1.06000	1.07000	1.08000	1.09000	1.10000	1
2	1.08160	1.10250	1.12360	1.14490	1.16640	1.18810	1.21000	2
3	1.12486	1.15763	1.19102	1.22504	1.25971	1.29503	1.33100	3
4	1.16986	1.21551	1.26248	1.31080	1.36049	1.41158	1.46410	4
5	1.21665	1.27628	1.33823	1.40255	1.46933	1.53862	1.61051	5
6	1.26532	1.34010	1.41852	1.50073	1.58687	1.67710	1.77156	6
7	1.31593	1.40710	1.50363	1.60578	1.71382	1.82804	1.94872	7
8	1.36857	1.47746	1.59385	1.71819	1.85093	1.99256	2.14359	8
9	1.42331	1.55133	1.68948	1.83846	1.99900	2.17189	2.35795	9
10	1.48024	1.62889	1.79085	1.96715	2.15892	2.36736	2.59374	10
11	1.53945	1.71034	1.89830	2.10485	2.33164	2.58043	2.85312	11
12	1.60103	1.79586	2.01220	2.25219	2.51817	2.81266	3.13843	12
13	1.66507	1.88565	2.13293	2.40985	2.71962	3.06580	3.45227	13
14	1.73168	1.97993	2.26090	2.57853	2.93719	3.34173	3.79750	14
15	1.80094	2.07893	2.39656	2.75903	3.17217	3.64248	4.17725	15

To find the value of $(1.04)^{10}$ look for the column headed 4% in the amount table and for the row with 10 in the left-hand column. The intersection of this column and row gives a value of 1.48024.

$$\therefore (1.04)^{10} = 1.48024$$
$$\text{Similarly, } (1.04)^{5} = 1.21665$$
$$\text{and } (1.04)^{12} = 1.60103$$

1. From the table find the following.
   (*a*) $(1.04)^6$ (*d*) $(1.10)^9$
   (*b*) $(1.07)^{11}$ (*e*) $(1.06)^{15}$
   (*c*) $(1.09)^{12}$ (*f*) $(1.08)^8$

Exercise 18.5

Written Solutions

1. Copy and complete the following table to show the amount of \$100 invested at 6% and 8%, interest compounded annually at the end of the years indicated.

End of Year	6%	8%
10		
11		
12		
15		
20		
30		

2. Find the following amounts.
   (*a*) $100(1.025)^{14}$ (*d*) $1000(1.035)^8$
   (*b*) $50(1.07)^{10}$ (*e*) $300(1.04)^{35}$
   (*c*) $25(1.015)^{40}$

3. Find the amount of each of the following.
   (*a*) \$400 invested at 6% for 3 years compounded annually
   (*b*) \$400 invested at 6% for 3 years compounded semi-annually
   (*c*) \$400 invested at 6% for 3 years compounded quarterly
   (*d*) \$400 invested at 6% for 3 years compounded monthly
   (*e*) \$500 invested at 10% for 5 years compounded quarterly
   (*f*) \$500 invested at 10% for 10 years compounded semi-annually
   (*g*) \$2000 invested at 6% for 8 years compounded annually

Use the compound interest tables.

2.(a) 141.30
(b) 98.36
(c) 45.35

3.(a) \$476.41
(b) \$477.62
(c) \$478.25
(d) \$478.67

NUMBER OF PERIODS

$$A = P(1 + i)^n$$

AMOUNT PRINCIPAL INTEREST PER PERIOD

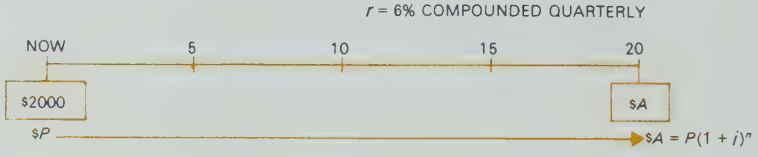
$(1.005)^{36} = 1.19668$

18.6 Problems Involving Amount

Example 1

Determine the amount of an investment of \$2000 invested at 6% compounded quarterly for 10 years.

Solution



$$i = \frac{r}{4} = \frac{0.06}{4} = 0.015 \text{ (interest per period)}$$

$$n = (10)(4) = 40 \text{ (number of periods)}$$

$$\therefore A = 2000(1.015)^{40}$$

$$= 2000(1.81402)$$

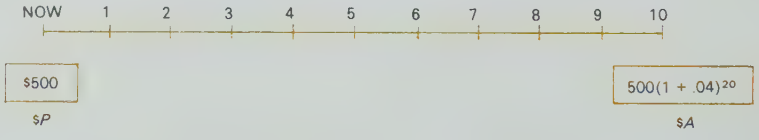
$$= 3628.04$$

$\therefore$ the investment will amount to \$3628.04.

Example 2

Melvin Moneymaker deposits a sum of \$500 in a savings account at 8% interest compounded semi-annually. How much does his deposit amount to at the end of 10 years?

Solution



$$A = P(1 + i)^n$$

where $P = 500$

$$i = \frac{0.08}{2} = 0.04 \text{ (interest rate per period)}$$

$$n = 10(2) = 20 \text{ (number of periods)}$$

$$A = 500(1 + 0.04)^{20}$$

$$= 500(2.19112)$$

$$= 1095.56$$

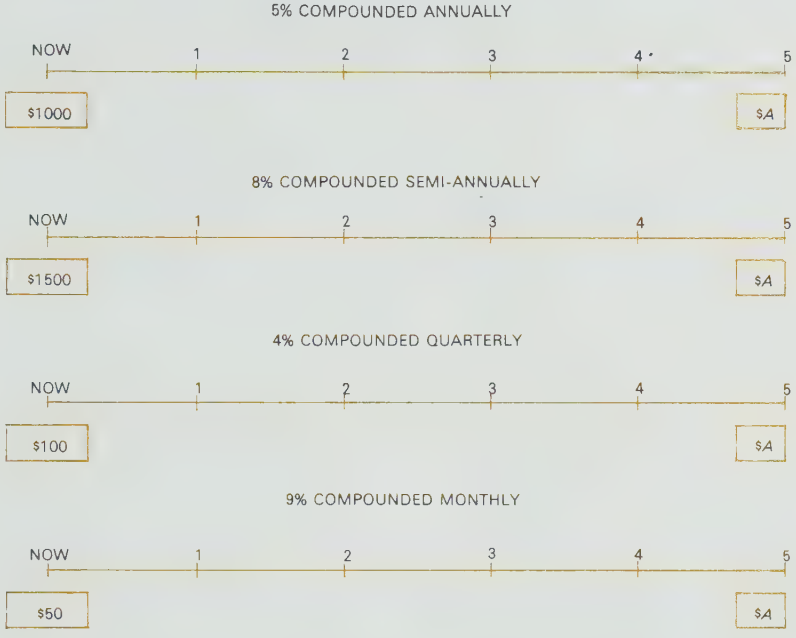
$\therefore$ there will be \$1095.56 in Mel's account at the end of 10 years.

Wow! That's more than double the principal!

Exercise 18.6

Discussion

1. For each of the following line diagrams state the values for P, i, and n.



Written Solutions

2. For each of the line diagrams in Question 1 state the amount at the time indicated.

3. Make a line diagram to indicate the amount for each of the following.
   (*a*) \$2500 invested for 5 years at 5% compounded annually
   (*b*) \$3000 invested for 8 years at 6% compounded semi-annually
   (*c*) \$5000 invested for 10 years at 8% compounded quarterly
   (*d*) \$100 invested for 40 years at 9% compounded annually
   (*e*) \$4800 invested for 20 years at 10% compounded semi-annually
   (*f*) \$3400 invested for 15 years at 4% compounded quarterly

$(1.0075)^{60} = 1.56568$

2.(a) \$1276.28
(b) \$2220.36

5. $1902.36

6.(b) $3022.14
(c) $2859.00

8. $30 652.05

10. $400.00

4. Copy and complete the following table.

Principal	*Rate*	*Time* (years)	*Compounded*	*Amount*
$100	9%	$5\frac{1}{2}$	semi-annually	
			annually	$500(1.04)^{14}$
$400	6%	2	quarterly	
			quarterly	$30(1.01)^{16}$
			monthly	$1000(1.0075)^{36}$

5. A man borrows $1500 from the bank at 8% per annum compounded quarterly. Find the amount he must pay in 36 months.

6. Orville Operator has the opportunity to invest $2000 for 6 years. He is considering three possible investments (*a*) 8% compounded annually, (*b*) 7% compounded semi-annually, and (*c*) 6% compounded quarterly. What would be the amount of each investment, and in terms of the dollar value, which is the best choice?

7. On the day his son was born, Sam Silverspoon invested $100 at 10% per annum compounded semi-annually to provide for the boy's education. How much will the boy have on his eighteenth birthday?

8. The Millicent M. Moneymaker Memorial Scholarship Fund was to be set up based on a gift of $15 000. If the money was invested at 6% compounded quarterly, how much money was available for scholarships at the end of the 12th year?

9. Phil Anthropist donated $25 000 to a building fund, stipulating that a new room should be added to the library in 10 years with the proceeds of the fund. If the fund earned 8% compounded annually, how much was available to build the extra room at the end of the 10th year?

10. A cash prize was invested at 4%, 40 years ago, for the solution of a problem. The problem is yet to be solved. If the prize now amounts to $1920.41, how much was the original prize?

11. Tim Titepenny decided to invest \$200 each August from his summer's earnings. Separate investments were made Sept. 1, 1976, Sept. 1, 1977, and Sept. 1, 1978 into an account paying 7% per annum compounded annually. If no withdrawals are made, how much will Tim have on Sept. 1, 1991?

12. When Grandmother Goldbrick died, she left \$14 000 and \$15 000, respectively, to her granddaughters Lynda and Debbie. If Lynda invested her money at 6%, and Debbie invested hers at 5%, what was the difference in their holdings after 12 years?

13. Larry Lookahead decided to invest a sum of money that would pay for a cottage on a northern lake at the time of his retirement in 8 years. Find the amount of the investment that would give him \$8500, if the sum was invested at 7% per annum compounded semi-annually.

14. When the well-drilling partnership between Jack and Jill was dissolved, Jack took over the company and in settlement gave Jill the following:
    (*a*) \$12 500 in cash,
    (*b*) a note for \$5000 payable in 5 years,
    (*c*) another note for \$7500 payable 8 years from the time of dissolution.

    Jill took the notes to a trust firm where they agreed to give her \$15 000 now. Find the total interest in dollars that was earned by the trust company. Money earns 10% per annum compounded quarterly. How much did Jill lose by taking the cash?

12. **\$3028.76**

14. **Total interest earned by Trust company was \$3954.62. Jill lost \$3954.62**

Unit 19

Applications of Percentage

19.1 Cost Price and Selling Price

A The prime cost or net cost of an article is the price appearing on the bill of invoice after all discounts have been deducted. In order to determine his selling price the merchant will mark up the cost price by a certain percentage. This relation can be written as follows:

$$\text{Selling price} = \text{prime cost} + \text{markup}$$

Example 1

What is the selling price of an article whose prime cost is \$10 if the markup is 40% of the cost?

Solution

$$\begin{aligned} \text{S.P.} &= 10 + 40\% \text{ of } 10 \\ &= 10 + (0.40)(10) \\ &= 10 + 4 \\ &= 14 \end{aligned}$$

$\therefore$ the selling price is \$14.00

Now try these.

1. Calculate the selling price of each of the following articles.
   (*a*) cost of a pair of shoes is \$25.00 with a markup of 50%
   (*b*) cost of a carton of cigarettes is \$12.50 with a markup of 70%
   (*c*) cost of a hockey stick is \$3.00 with a markup of 40%
2. Calculate the markup on each of the following quantities.
   (*a*) the cost price of a sweater is \$12.00 and the selling price is \$15.00
   (*b*) the selling price of a TV set is \$499.00 and the cost price is \$375.00
3. What is the percent markup in each of the following?
   (*a*) the selling price of a pair of jeans is \$9.00 and the cost price is \$6.00
   (*b*) the cost price of a pair of skis is \$28.00 and the selling price is \$46.00

B The prime cost of an article does not make up all the costs to a merchant. He has other expenses such as light, heat, rent, insurance, sales help, and office expenses. These charges make up his *overhead*. A certain percentage of this overhead must be included in the gross cost of each article. This relation can be written as follows:

$$\boxed{\text{Gross cost} = \text{prime cost} + \text{overhead}}$$

It follows that if a merchant is going to make a profit this must be included in the markup. Therefore we can say

$$\boxed{\text{Markup} = \text{overhead} + \text{profit}}$$

Example 1

If the overhead of a store is considered at 30% of the prime cost, calculate the overhead, gross cost, markup, and profit of an article whose prime cost is \$20 and which has a markup rate of 50%.

Solution

$$\begin{aligned}
\text{Overhead} &= 30\% \text{ of } 20 \\
&= (0.30)(20) \\
&= 6 \\
\text{Gross cost} &= 20 + 6 \\
&= 26 \\
\text{Markup} &= 50\% \text{ of } 20 \\
&= (0.50)(20) \\
&= 10 \\
\text{Profit} &= \text{Markup} - \text{overhead} \\
&= 10 - 6 \\
&= 4
\end{aligned}$$

$\therefore$ Overhead is \$6
Gross cost is \$26
Markup is \$10
Profit is \$4

See if you can do these.

1. If the overhead of a bookstore is calculated to be 10% of the cost of each book sold, calculate the overhead, gross cost, markup, and profit on a book which costs \$8.00 and has a markup of 25%.
2. Calculate the markup that an appliance store charges on a radio whose prime cost is \$21. The overhead is calculated on the basis of 33% of the cost and the dealer wants to make \$10 on each radio sold.

Summary

$$\begin{aligned}
\text{Selling price} &= \text{prime cost} + \text{markup} \\
\text{Gross cost} &= \text{prime cost} + \text{overhead} \\
\text{Markup} &= \text{overhead} + \text{profit} \\
\text{Selling price} &= \text{prime cost} + \text{overhead} + \text{profit}
\end{aligned}$$

Exercise 19.1

Discussion

1. How can a dealer increase his profit if he has a fixed prime cost?
2. What increase in costs will increase the overhead of a company?
3. What is the effect of an increasing overhead in a company?
4. How could a company maintain a fixed selling price but increase profits?
5. In order to establish a markup what are some of the items a merchant must consider?

Written Solutions

6. What is the selling price of an article, if the markup is 40% of the prime cost of \$7.60?
7. What is the percent markup on a pair of hockey skates that sell for \$24.95 if the prime cost is \$19.95?
8. The prime cost of an article is \$120. If the overhead is 25% of the prime cost, what is the gross cost?
9. The prime cost of a stereo set is \$720 and the overhead is calculated at 30% of the cost.
   (*a*) What is the markup if the selling price is \$1200?
   (*b*) What is the overhead?
   (*c*) What is the profit?
10. The selling price of a blender is \$33.
    (*a*) If the rate of markup is 40% of the prime cost, what is the prime cost?
    (*b*) Calculate the markup.
    (*c*) Calculate the profit if the overhead is 50% of the prime cost.
11. A dealer's prime cost for an electric light fixture is \$24.00.
    (*a*) What is the price to the consumer if the markup is 30%?
    (*b*) If the overhead is two-thirds of the markup, what is the profit?

6. \$10.64
8. \$150
10. (a) \$23.57
    (b) \$9.43
    (c) *Loss* is \$2.36

12. Copy and complete the following table in your book.

Item	*Selling Price*	*Prime Cost*	*Overhead*	*Gross Cost*	*Markup*	*Profit*
1	$1500	$950	$250			
2	$45	$25.50		$33		
3		$68.50			$56.50	$23.00
4		$618		$837.50		$215
5	$70.20		$6.25			$4.80
6	$225		$29	$172		
7		$17.40			$4.30	$2.20
8		$1236		$1675		$430

19.2 Profit and Loss

The profit or loss is the difference between the selling price and the gross cost price. This statement can be shown by the following equation.

$$\text{Profit or loss} = \text{selling price} - \text{gross cost price}$$

In business, a very important statistic is the company's rate of profit. This rate is usually calculated on the cost price.

$$\text{Rate of profit} = \frac{\text{profit}}{\text{gross cost price}} \times 100$$

Example 1

Find the rate of profit to one decimal when an article costing $241.00 sells for $276.00.

Solution

$$\begin{aligned} \text{Profit} &= \text{S.P.} - \text{C.P.} \\ &= \$276.00 - \$241.00 \\ &= \$35.00 \\ \therefore \text{rate of profit} &= \frac{35}{241} \times 100 \\ &= 14.5\% \end{aligned}$$

Example 2

With a profit of 20%, an article costs \$276.00. Find the selling price.

Solution

$$\text{Rate of profit} = \frac{\text{profit}}{\text{cost price}} \times 100$$

$$\therefore \frac{20}{100} = \frac{\text{profit}}{\$276.00}$$

$$\text{Profit} = \frac{20 \times 276}{100}$$

$$= \$55.20$$

$$\text{Profit} = \text{selling price} - \text{cost price}$$

$$\therefore \text{selling price} = \text{profit} + \text{cost price}$$

$$= \$55.20 + \$276.00$$

$$= \$331.20$$

Exercise 19.2

Written Solutions

Profit = SP − CP

Rate of Profit = $\frac{\text{Profit}}{\text{CP}} \times 100$

1. Find the rate of profit to the first decimal.

	Cost Price	*Selling Price*
(*a*)	\$97.50	\$117.50
(*b*)	\$165.85	\$182.15
(*c*)	\$14.20	\$18.40
(*d*)	\$37.25	\$52.50
(*e*)	\$256.50	\$286.15

2. Find the selling price.

	Cost Price	*Rate of Profit*
(*a*)	\$760.00	15%
(*b*)	\$210.00	11%
(*c*)	\$375.00	20%
(*d*)	\$96.00	8%
(*e*)	\$2416.00	25%

1.(d) 40.9%
(e) 11.6%
2.(d) \$103.68
(e) \$3020.00

3. Find the cost price.

	Selling Price	*Rate of Profit*
(*a*)	\$1200.00	9%
(*b*)	\$760.00	5%
(*c*)	\$270.00	15%
(*d*)	\$824.00	12%
(*e*)	\$925.00	$33\frac{1}{3}$%

4. Find the selling price.

	Cost Price	*Rate of Loss*
(*a*)	\$185.00	12%
(*b*)	\$75.70	25%
(*c*)	\$210.00	8%
(*d*)	\$95.80	20%
(*e*)	\$2020.00	15%

5. Eggs cost a farmer 52 cents a dozen. They are sold at a profit of 25%. Find the selling price.

6. The cost of creamery butter is 40 kg for \$75.00. Calculate the selling price if a profit of 15% is to be made.

7. A furniture dealer pays \$75.00 for a chair. At what price does he sell the chair to realize a 25% profit?

8. A furniture dealer pays \$125.00 for a chair. His selling expenses are 15% of the cost price. At what price does he sell the chair to realize a 20% profit?

9. A house is bought for \$40 000. The expense to convert the house into a duplex is estimated at 15% of the cost. If a profit of 12% is desired, at what price should the house be sold?

10. Silver dollars increase in value over a period of years. Calculate the rate of profit for each coin dated as follows.

Year	*Cost Price*	*Selling Price*
1935	\$2.00	\$3.50
1938	\$4.50	\$7.50
1945	\$15.00	\$27.50
1951	\$1.25	\$2.60
1961	\$1.50	\$3.25
1966	\$2.50	\$3.10

Careful here!

And here!

3.(d) \$735.71
(e) \$693.75

4.(d) \$76.64
(e) \$1717.00

6. \$86.25

7. \$93.75

8. \$172.50

19.3 Trade Discounts

A Most wholesalers or manufacturers offer quantity discount to encourage customers to buy a large supply of a particular product.

Example 1

Calculate the price a retailer must pay for 100 packages of fireworks at \$0.95 per package less 10%.

Solution

$$\begin{aligned} \text{List Price} - \text{discount} &= \text{amount owing} \\ 100(\$0.95) - (10\% \text{ of } \$95.00) &= \text{amount owing} \\ \therefore \text{amount owing} &= \$95.00 - \$9.50 \\ &= \$85.50 \end{aligned}$$

Two discounts are allowed - 10% followed by 5%.

Example 2

How much must a retail food producer pay for 15 bags of onions at \$1.75 per bag less 10%, 5%?

Solution

$$\begin{aligned} \text{List price} - \text{first discount} &= \text{first amount owing} \\ \text{First amount owing} - \text{second discount} &= \text{amount owing} \\ \therefore 15(1.75) - 10\% \text{ of } 15(1.75) &= \text{first amount owing} \\ \therefore (26.25 - 2.63) - 5\% \text{ of } \$23.62 &= \text{amount owing} \\ \therefore \text{amount owing} &= \$23.62 - 1.18 \\ &= \$22.44 \end{aligned}$$

B Often the terms of a bill will allow special discounts if it is paid within a specified period. Some of the terms of a bill may be as follows.

(*a*)	Cash:	The bill must be paid immediately.
(*b*)	N/30:	The full amount of the bill is to be paid within 30 days.
(*c*)	2/10:	A discount of 2% is allowed if payment is made within 10 days of the billing date.
(*d*)	2/10, N/30:	A discount of 2% is allowed if payment is made within 10 days of the billing date, otherwise the full amount is due in 30 days.

Example 1

The terms of an invoice for \$210 dated July 4 are 2/15, N/30. If the bill is paid on July 14, how much should be paid?

Solution

Since July 14 is within 15 days of the billing date, a 2% discount can be applied.

$$\therefore \$210.00 - 2\% \text{ of } \$210 = \text{amount owed}$$
$$\therefore \text{amount owed} = \$210.00 - \$4.20$$
$$= \$205.80$$

Exercise 19.3

Written Solutions

1. State the date by which the following should be paid:
   (i) to receive a discount
   (ii) to fulfill the terms of the invoice

	Invoice Date	*Terms*
(*a*)	April 4	2/10, N/15
(*b*)	June 11	3/15, N/30
(*c*)	Aug. 17	5/10, N/60
(*d*)	Sept. 21	N/60
(*e*)	Dec. 2	3/15, N/30

2. Find the amount that would be paid to obtain the best possible price of an article costing \$100.
   (*a*) discounts of 20% and 10% or a discount of 30%
   (*b*) discounts of 15%, 10% and 5% or discounts of 20% and 5%
   (*c*) discounts of $12\frac{1}{2}\%$ and $7\frac{1}{2}\%$ or discounts of 15% and 5%

3. Calculate the amount owing in each of the following cases.

	List Price	*Trade Discount*
(*a*)	\$107.25	$33\frac{1}{3}\%$
(*b*)	\$78.22	15%, 10%
(*c*)	\$706.74	20%, 10%
(*d*)	\$238.02	15%, 10%, 5%
(*e*)	\$2169.52	42%

1.(a) (i) April 14
(ii) April 19
(c) (i) August 27
(ii) October 17

2.(a) \$70.00

3.(a) \$71.50
(d) \$172.98

4. Find the amount that would be paid on each of the following if the discount is obtained in each case.

	Amount of Invoice	*Terms*
(*a*)	\$87.34	2/10, N/30
(*b*)	\$546.08	2/10, N/15
(*c*)	\$112.59	3/30, N/90
(*d*)	\$642.98	3/10, 2/15, N/30
(*e*)	\$1525.86	3/15, N/30

5. An invoice dated March 14 for \$568.74 has terms of 3/15, 2/30, N/90. You pay 40% on the first discount date, 40% on the second discount date, and the remaining portion of the invoice on the final date. Find
   (*a*) each date of payment,
   (*b*) the payment made on each date,
   (*c*) the total amount paid.

6. On April 20 you receive an invoice bearing the date April 18 for \$493.15. The terms are 2/10, N/30. If you pay 50% of the invoice on the first discount date and the remainder on the final date, find the following.
   (*a*) each payment date
   (*b*) the payment made on each date
   (*c*) the total amount paid

7. A shipment of goods bears an invoice dated October 10 amounting to \$1061.34 and has terms of 3/10, N/30. On the first discount date you pay 60% of the invoice and on the final payment date you pay the remaining amount. Find the following.
   (*a*) each date of payment
   (*b*) the amount of each payment
   (*c*) the total amount paid

4.(a) \$85.59
(d) \$623.69

5.(b) \$220.67, \$222.95, \$113.74

6.(c) \$488.22

7.(a) October 20, November 9

19.4 Commission

A company that sells goods through agents usually pays these agents a *commission*. This commission is based on a percentage of the selling price of the goods.

A *Straight Commission*

A salesman who works on straight commission receives an income that is a stated percentage of the sales he makes.

Example

A furniture salesman works on a straight commission of 8% of his sales. If his total sales for the week amount to \$1500, calculate his commission.

Solution

$$\begin{aligned}\text{Commission} &= \text{rate} \times \text{sales} \\ &= 8\% \text{ of } \$1500 \\ &= \frac{8}{100} \times \$1500 \\ &= \$120.00\end{aligned}$$

$\therefore$ the salesman earns \$120.00 commission.

Try these.

1. A car salesman earns a straight commission of 4% of his sales. If his total sales for the month amount to \$30 000, calculate his month's commission.
2. A heavy equipment salesman earns 10% straight commission on his sales. What will be his total sales if his salary for the year was \$15 000?

B *Salary Plus Commission*

A salesman who is paid salary plus commission receives a basic salary for a specific time interval plus a certain percentage of his total sales.

Example

A used car salesman is paid a salary of \$90.00 per month plus 9% of his total sales for the month. Calculate his earnings for the month when his sales totalled \$12 825.00.

Solution

$$\begin{aligned}\text{Salary} &= \text{fixed salary} + \text{commission} \\ &= \$90.00 + (9\% \text{ of } \$12\,825.00) \\ &= \$90.00 + \$1154.25 \\ &= \$1244.25\end{aligned}$$

$\therefore$ the salesman earns \$1244.25 for the month.

1. A sporting goods salesman earns a salary of \$100 a month plus 6% of his total sales for the month. Calculate his earnings for the month based on sales of \$15 000 for the month.

2. A real estate agent earns \$200 per month plus 1% of his total sales. If a salesman sells a house worth \$75 000 during December, calculate his December commission.

C *Graduated Commission*

Often companies will pay a well established salesman a graduated commission in order to provide greater incentive for sales over a stated amount.

Example

Thomas Edison of Light Incorporated is paid on the following basis: 5% on his first \$2500 of sales each month, 8% on his next \$6500 of sales and 10% on all sales over \$9000. If his sales for the month amount to \$13 250, calculate his income for the month.

Solution

$$\begin{aligned}\text{Income} &= 5\% \text{ of } \$2500 + 8\% \text{ of } \$6500 + 10\% \text{ of } \$4250\\ &= \$125 + \$520 + \$425\\ &= \$1070\end{aligned}$$

$\therefore$ the salesman's income for the month is \$1070.00

Exercise 19.4

Written Solutions

1. Find the commission based on the following information.

	Sales	*Rate of Commission*
(*a*)	\$1750	20%
(*b*)	\$955	4%
(*c*)	\$25 000	$\frac{1}{2}$%
(*d*)	\$20 000	2%
(*e*)	\$2505	$3\frac{1}{2}$%

1.(a) \$350.00
(b) \$38.20
(c) \$125.00

2. What is the rate of commission earned in each of the following?

	Sales	*Commission*
(*a*)	\$4344	\$362
(*b*)	\$2400	\$720
(*c*)	\$5250	\$315
(*d*)	\$896	\$112
(*e*)	\$18 000	\$180

3. The following salesmen are paid straight commission on their sales. Calculate the commission earned by each salesman for the period shown.
   (*a*) $2\frac{1}{4}\%$ of sales on \$13 900 in a week
   (*b*) $\frac{1}{2}$ of 1% of sales on \$290 250 in a month
   (*c*) $2\frac{1}{2}\%$ of sales on \$11 450 in a week

4. The following salesmen are paid a salary plus commission. Calculate the gross pay for each agent for the period shown.
   (*a*) \$50 per week plus 2% of sales of \$10 280 in one week
   (*b*) \$70 per week plus $\frac{1}{2}\%$ of over \$4000 on sales of \$28 000 in a week
   (*c*) \$150 per month plus 2% of sales of \$74 000 in one month
   (*d*) \$160 per month plus $1\frac{1}{4}\%$ of \$79 076 in sales in one month
   (*e*) \$80 per month plus 12% of \$8048 in sales in one month

5. A salesman earns a basic salary of \$150 weekly plus 2% on sales between \$1000 and \$2000 plus 8% on sales over \$2000. If his records show \$2505 in sales for a one week period, calculate his earnings for that period.

6. A salesman sent to his company the proceeds of his sales for a one week period. He has deducted his salary of \$90, his expenses of \$185, and his commission of 10% of sales. The amount of the cheque was \$2110. What were his total sales?

7. Find the total commission received in the following cases.

	Sales	*Rate on First \$500*	*Rate on Second \$500*	*Rate on Remainder*
(*a*)	\$730	4%	5%	6%
(*b*)	\$206	8%	6%	4%
(*c*)	\$1520	5%	7%	9%
(*d*)	\$5632	2%	3%	4%
(*e*)	\$1090	$\frac{1}{2}\%$	2%	8%

2.(a) 8.3%
(b) 30%
(c) 6%

3.(a) \$312.75

4.(a) \$255.60
(d) \$1148.45

6. \$2650

7.(a) \$31.50
(d) \$210.28

Unit 20

Perimeter, Area, and Volume

Introduction

In this unit we will be calculating distances, areas, and volumes. It will be useful to remember the following metric conventions.

1. No period is needed after the symbol. Write 10 metres as 10 m not 10 m., unless it ends a sentence.
2. Use the same symbol for singular and plural.
3. Most symbols use small letters.
4. Units of area are cm^2, m^2, and volume cm^3, m^3.
5. Use a space instead of a comma for numbers of more than four digits, such as 20 000 m or 0.000 82 mm.

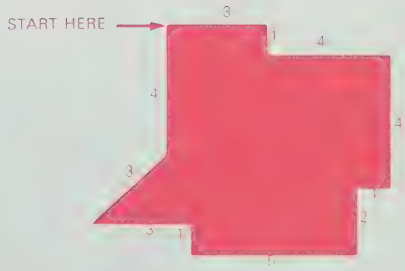
We will use the following everyday metric units.

For length:		For area:	
millimetre	mm	square centimetre	cm^2
centimetre	cm	square metre	m^2
metre	m	hectare (10 000 m^2)	ha
kilometre	km	square kilometre	km^2

For volume:		Conversion factors:
cubic centimetre	cm^3	10 mm = 1 cm
litre (1000 cm^3)	litre or ℓ	100 cm = 1 m
kilolitre	kl	1000 m = 1 km

20.1 Perimeter

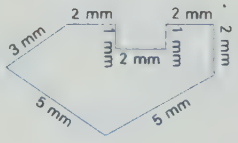
Perimeter is the distance measured around a figure.



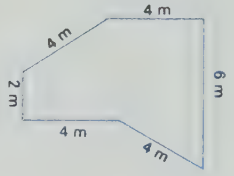
Perimeter = 3 + 1 + 4 + 4 + 1 + 2 + 5 + 1 + 3 + 3 + 4
= 31 units

1. Determine the perimeter of each figure.

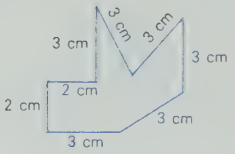
(a)



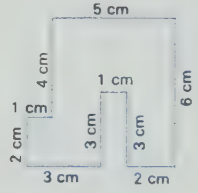
(c)



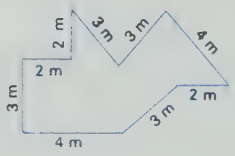
(e)



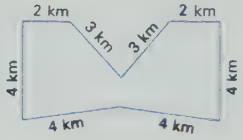
(b)



(d)

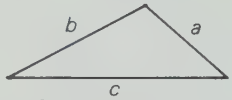


(f)



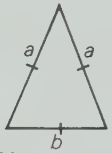
The perimeters of some familiar plane figures are given below.

"Plane" means "flat"



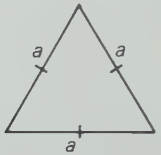
SCALENE TRIANGLE

$P = a + b + c$



ISOSCELES TRIANGLE

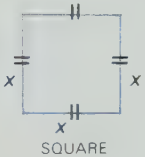
$P = 2a + b$



EQUILATERAL TRIANGLE

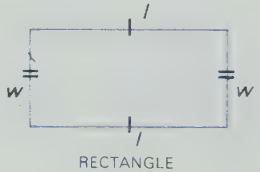
$P = 3a$

The perimeter of a circle is called the circumference



SQUARE

$P = x + x + x + x$
$= 4x$

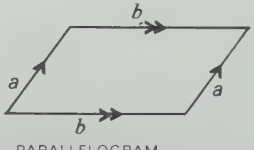


RECTANGLE

$P = l + w + l + w$
$= 2l + 2w$
$= 2\,(l + w)$

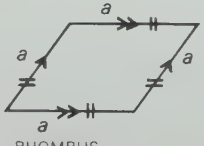
CIRCLE

$P = 2\pi r$



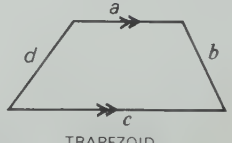
PARALLELOGRAM

$P = a + b + a + b$
$= 2a + 2b$
$= 2(a + b)$



RHOMBUS

$P = a + a + a + a$
$= 4a$



TRAPEZOID

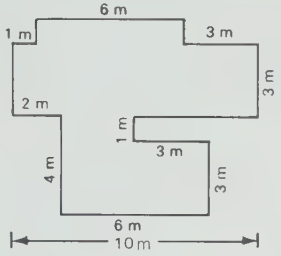
$P = a + b + c + d$

Exercise 20.1

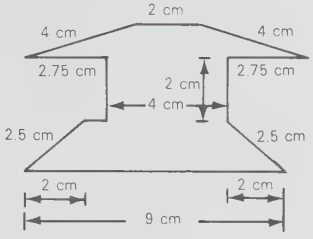
Written Solutions

1. Determine the perimeters of each figure.

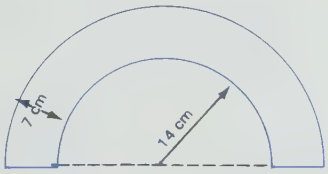
(*a*)



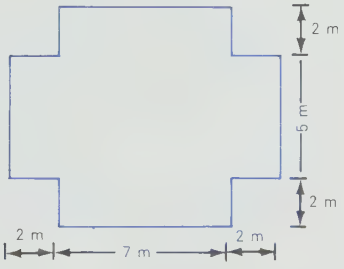
(*c*)



(*b*)



(*d*)



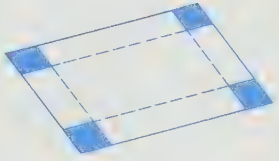
2. A topless cardboard carton is to be formed by cutting squares from the four corners of a rectangular section of cardboard and folding up the flaps formed. If the bottom of the finished carton is to be 10 cm × 12 cm and the height is to be 6 cm, what are the dimensions of the sheet of cardboard? What is its perimeter?

3. Calculate the perimeter of each of the following figures.
   (*a*) a circle radius 28 cm
   (*b*) a rectangle 15 m × 45 m
   (*c*) a parallelogram having sides 6 cm and 10 cm
   (*d*) a trapezoid with parallel sides of 3 m and 2 m, the other two sides being 7 m and 6 m
   (*e*) an equilateral triangle of side 2.5 cm
   (*f*) a circle of diameter 63 mm

4. A poker table is to be made from a sheet of plywood. The table is to be in the shape of a regular octagon (eight sides equal). If each side is 45 cm, how much metal edging will be necessary for the table?

5. What are the dimensions of a square of perimeter 8.32 m?

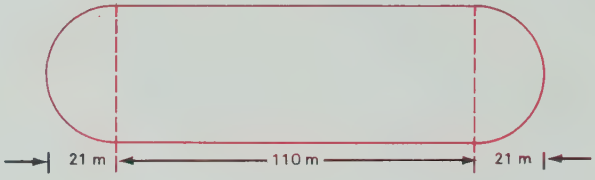
$\pi \doteq \frac{22}{7}$



1.(a) 42 m
(b) 124 cm
3.(a) 176 cm
(d) 18 m
4. 360 cm

6. If the length of a rectangle is 2 m more than the width, what are the dimensions if the perimeter is 44 m?

7. Find the length of fencing required to enclose a field consisting of a rectangle with semicircular ends as shown. ($\pi \doteq \frac{22}{7}$)

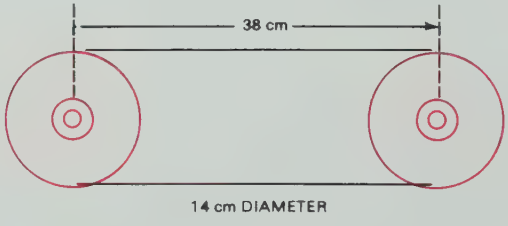


8. Find the length of a steel band required to go around a pressure tank that has an average diameter of 1.98 m. (Allow 35 cm extra for fastening.) State your answer in metres, correct to the nearest centimetre. ($\pi \doteq 3.142$)

9. A coil of wire has an average diameter of 1.82 m. How many metres are contained in the coil if there are 43 turns in all? ($\pi \doteq \frac{22}{7}$)

10. A circular field for testing model airplanes has a circumference of 300 m. Find the radius to the nearest tenth of a metre. (Use $\pi \doteq 3.14$.)

11. Find the length of belt required to connect the pulleys shown. ($\pi \doteq \frac{22}{7}$)



12. Determine the amount of floor moulding required for a room that measures 2.59 m by 4.32 m. Disregard any allowance for doors.

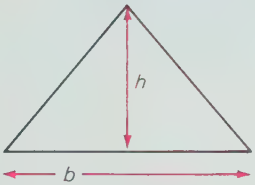
13. A triangle has a perimeter of 14.6 m. If two of the sides measure 6.5 m and 4.0 m, how long is the third side?

14. Find the length of metal edging to the nearest metre that would be required to go around a square table of measure 1.74 m to the side.

6. 10 m × 12 m
8. 6.57 m
10. 47.8 m
12. 13.82 m
14. 7 m

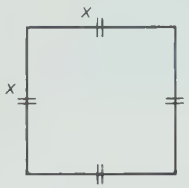
20.2 Area

If you cut around the perimeter of a plane figure, the region covered by the cutout is called the *area* of the figure. The areas of the more familiar figures are given below.



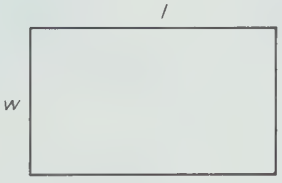
TRIANGLE

$A = \frac{1}{2}bh$



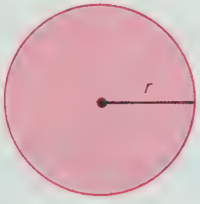
SQUARE

$A = x^2$



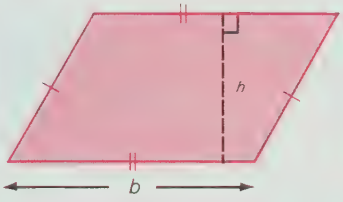
RECTANGLE

$A = lw$



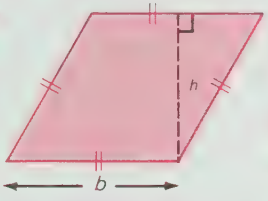
CIRCLE

$A = \pi r^2$



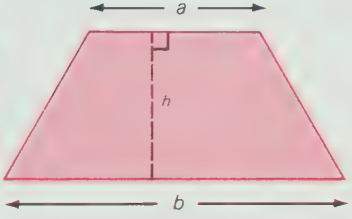
PARALLELOGRAM

$A = bh$



RHOMBUS

$A = bh$



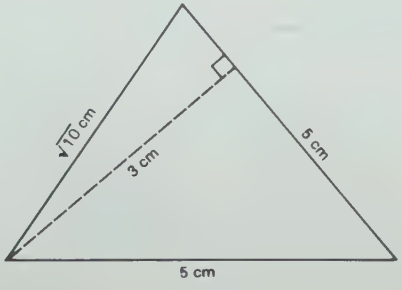
TRAPEZOID

$A = h\left(\frac{a+b}{2}\right)$

Examples

Calculate the area of each plane figure.

Solution (*a*)



$$A = \frac{1}{2}bh$$

"b" represents any side.

$$A = \frac{1}{2} \times 5 \times 3$$

but "h" represents the height drawn to the side used.

$$= \frac{15}{2}$$

Why?

$\therefore$ area $= 7.5\ \text{cm}^2$

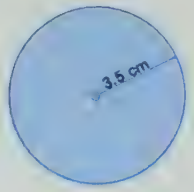
Solution (b)

$$A = \pi r^2$$

$$= \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}$$

$$= \frac{77}{2}$$

$\therefore$ area $= 38.5 \text{ cm}^2$



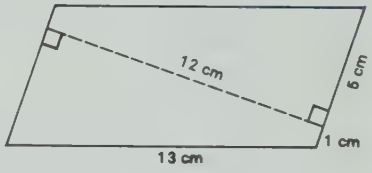
"b" represents any side, but "h" represents the height drawn to the side used.

Solution (c)

$$A = bh$$

$$= 6 \times 12$$

$\therefore$ area $= 72 \text{ cm}^2$



The distance between the parallel sides a and b.

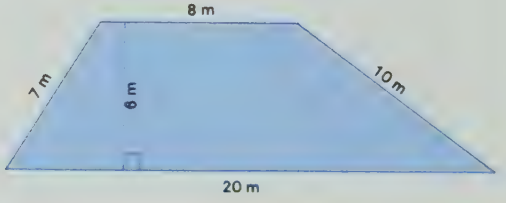
Solution (d)

$$A = \frac{h(a+b)}{2}$$

$$= \frac{6(8+20)}{2}$$

$$= 3(28)$$

$\therefore$ area $= 84 \text{ m}^2$



Exercise 20.2

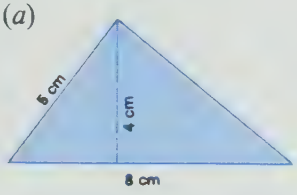
Discussion

1. For a triangle, $A = \frac{1}{2}bh$. In this formula, what do b and h represent for a triangle?
2. What do you mean by the "height" of a triangle? How many different heights are there for any scalene triangle?
3. What is the minimum amount of information you would need in order to calculate the area of a parallelogram? How many different heights are there for any given parallelogram?
4. In calculating the area of a trapezoid, what distances do you require? What distances do you need?

Written Solutions

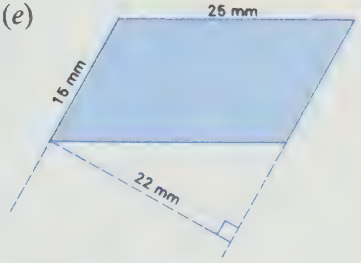
5. Calculate the area of each figure.

(*a*)

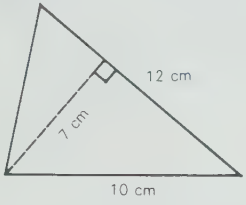


(*c*)

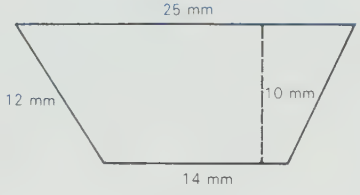
(*e*)



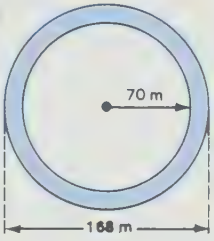
(*b*)



(*d*)

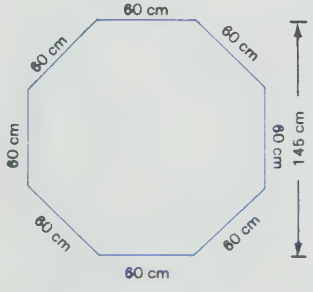


(*f*)

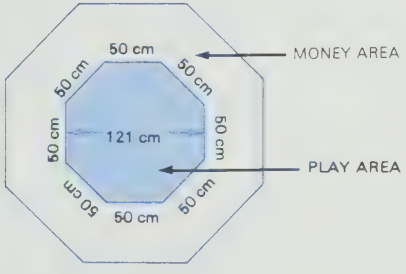


6. Here's that poker table again! Calculate the total area in (*a*). Calculate the area of felt required for the centre part in (*b*). How much area is available for each player's money?

(*a*)

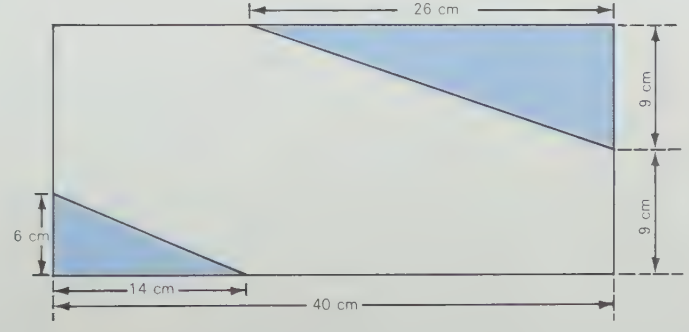


(*b*)



$\pi = \frac{22}{7}$

7. Find the area of the two shaded triangles shown in the diagram. What is the area of the remaining unshaded section?

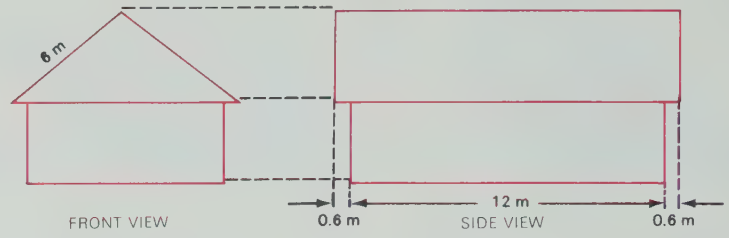


5.(a) 16 cm²
(b) 42 cm²

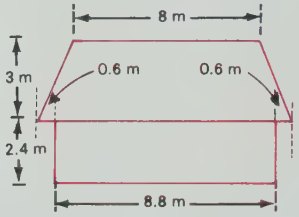
6.(a) 1.74 m²

7. Area of triangles = 42 cm² and 117 cm²

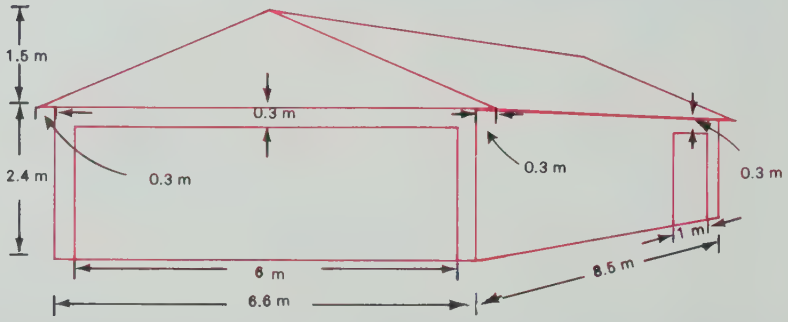
8. The roof shown below is to be shingled. Find its area.



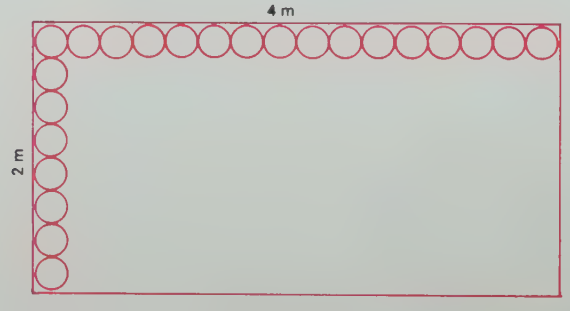
9. A Tudor-style house has a roof end as shown. Calculate the total area of shingles required to cover both roof ends.



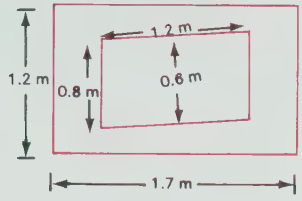
10. If 1ℓ of paint costs \$1.75 and covers 18 m^2, calculate the cost to paint the garage shown, not including the large door.



11. Circular discs 25 cm in diameter are stamped out of a sheet of metal 2 m $\times$ 4 m. What is the area of waste material? What is the per cent of waste material?

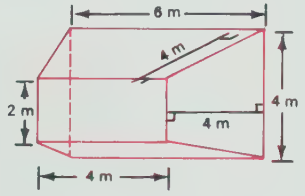


12. A parallelogram is punched out of a rectangular piece of sheet metal. Calculate the area of the parallelogram and the area of the remaining piece of sheet metal.



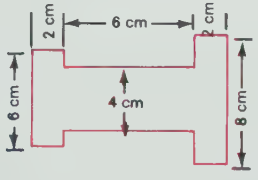
13. Two triangular pieces are cut from a 1.60 m square piece of tin. One triangle has a base of 72 cm and an altitude of 41 cm, while the other has a base of 45 cm and an altitude of 26 cm. Find the area of each triangular piece and the area of material left.

14. An open sand hopper is constructed from four trapezoidal pieces of sheet steel and a small rectangular piece as shown. Making no allowance for lapping, determine the minimum amount of sheet steel required.

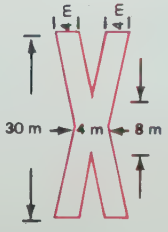


15. Calculate the area of each irregular figure.

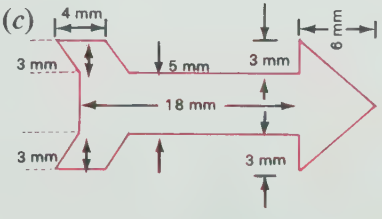
(*a*)



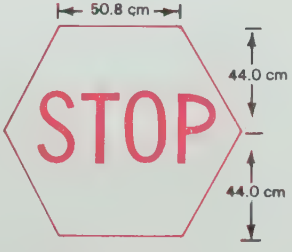
(*b*)



(*c*)



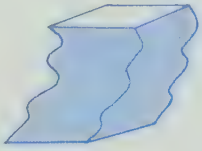
(*d*)



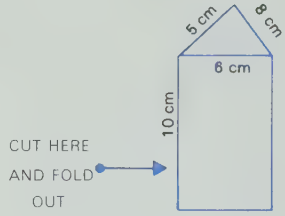
8. 158.4 m²
10. $7.00
12. 0.72 m², 1.32 m²
14. 72 m²
15.(a) 52 cm²
(b) 224 m²

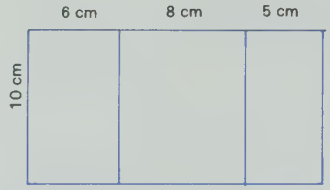
20.3 Surface Area of Right Prisms

A "right" prism has sides at right angles to the base.



A "WRONG" PRISM?



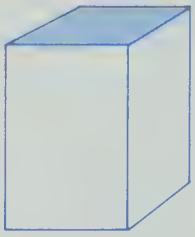


"S" MEANS "AREA OF SIDES"

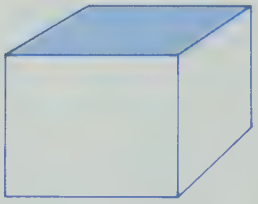
NOTE THAT THIS IS THE PERIMETER OF THE BASE

$$S = (6 + 8 + 5) \times 10$$
$$= 19 \times 10$$
$$\therefore \text{ area of sides} = 190 \text{ cm}^2$$

CIRCULAR PRISM (CYLINDER)



SQUARE PRISM



RECTANGULAR PRISM

"p" is the perimeter of the base, and "h" is the height.

For all prisms $S = ph$

The total surface area of a right prism would have to include the area of the top and the bottom. These areas would be calculated as in Section 20.2

$S = ph$

Exercise 20.3

1. Describe in words the method of calculating the surface area of the sides of a right prism.
2. Find the area of the sides of each right prism.
   (*a*) rectangular, 18 cm × 12 cm, height 6 cm
   (*b*) circular, radius 35 cm, height 8 cm
   (*c*) triangular, sides 2 cm, 5 cm, 6 cm, height 14 cm
   (*d*) regular octagon with sides 2 cm, height 10 cm
3. A can that holds 225 ml of liquid is 10 cm high with a diameter of approximately 5.37 cm. Calculate the amount of material required to make this can.

4. If a manufacturer were to make the can in Question 3 in the shape of a square prism, the dimensions for a can of equal capacity would be approximately 4.74 cm × 4.74 cm × 10 cm. Calculate the amount of material required to make the can this shape. Can you see the advantage to the manufacturer of the cylindrical shape over the square prism?

Be sure to include the ends.

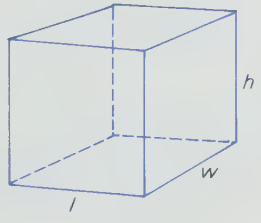
5. Find the cost of painting the walls of a room 4 m × 5 m × 2.5 m if 4ℓ of paint cost $7.29 and cover 90 m². Ignore openings for doors but allow for two coats of paint.

6. A circular tin pipe 2 m long is to be made from a sheet of tin 1.2 m × 2 m. What would be the diameter of the largest possible circular pipe? (Ignore overlap.)

7. Find the maximum number of sections of 15 cm stove pipe that can be cut from a sheet of tin 120 cm × 240 cm if each section is 60 cm long. Allow 1 cm for overlap. How much material is wasted?

2.(a) 360 cm²
(b) 1760 cm²
3. 214 cm²
4. 235 cm²
6. 38.2 cm

20.4 Volume of Right Prisms



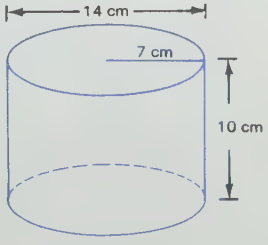
$$V = l \times w \times h$$
But $l \times w = A$
$$\therefore V = Ah$$

$V=Ah$

This formula can be used to calculate the volume of any right prism. Just be careful to calculate the area of the base with the appropriate formula.

Example

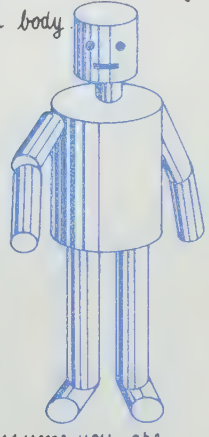
Calculate the volume of the given cylinder.



Solution

$$V = Ah$$
and here $A = \pi r^2$
$$= \frac{22}{7} \times 7 \times 7$$
$$= 154$$
$$\therefore V = 154 \times 10$$
$$\therefore \text{volume} = 1540 \text{ cm}^3$$

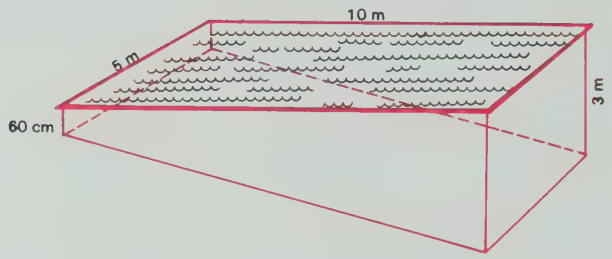
Find the volume of your own body.



(Assume you are composed of neat shapely cylinders.)

Exercise 20.4

1. Describe how you calculate the volume of *all* right prisms.
2. Calculate the volume of each prism described below.
   (*a*) rectangular, 6 cm × 8 cm, height 10 cm
   (*b*) square, 4 cm × 4 cm, height 1.5 cm
   (*c*) circular, diameter 3 m, height 8 m
   (*d*) parallelogram, $b = 4$ m, $h = 10$ m, height 1.5 m
   (*e*) triangular, $b = 7.5$ m, $h = 4.6$ m, height 1.8 m
3. Calculate the volume of earth that would have to be removed for the swimming pool shown. How many truck loads would be needed if the truck holds 4 m^3?



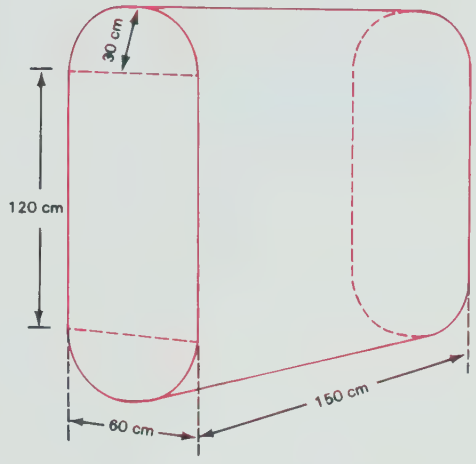
$1\ m^3 = 1\ kl$

4. When the swimming pool in Question 3 is filled with water, how many kilolitres will be required?
5. Take 3 sheets of paper of the same size and use them as follows.
   (*a*) Fold one into a cylinder.
   (*b*) Fold one into a square prism of the same height.
   (*c*) Fold one into a rectangular prism of the same height.

   Determine the necessary information in order to calculate
   (*d*) the lateral area of each,
   (*e*) the volume of each.

   Discuss your results.
6. The length of a gasoline storage tank is 65 cm, the width 49 cm, and the height 43 cm. Compute the capacity in litres.

7. A typical home oil storage tank is shown. Calculate its capacity in litres.



8. How high would a 25 ℓ pail have to be if the diameter is to be 30 cm?

9. Otto Basher wishes to build a garage having dimensions 7 m $\times$ 8.5 m. Before starting construction, he must pour a concrete floor 15 cm deep for the garage. How many "cubes" (cubic metres) of concrete will he have to order? How much will this cost at \$30/m^3?

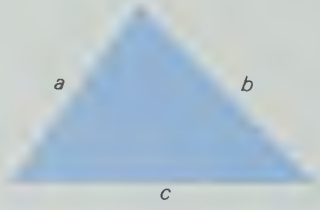
10. A railroad car used for shipping wheat to the West coast has inside dimensions of 12 m $\times$ 2.7 m $\times$ 3 m. If a tonne of wheat occupies 1.34 m^3, how many tonnes of wheat can be carried in each car? How many cars would be required to ship an order for 50 000 t?

11. How much earth is there in a cylindrical hole in the ground which is 9 m in diameter and 3 m deep?

2.(a) 480 cm^3
(b) 24 cm^3
3. 90 m^3
6. 137 ℓ
8. 35.4 cm
10. Approximately 72.5 t, 690 cars.

Area of Triangles

There is a formula that will determine the area of a triangle given only the lengths of the sides.



$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{a+b+c}{2}$ (or semi-perimeter)

Example 1

Find the area of the equilateral triangle shown.

Solution

$a = 6$ cm
$b = 6$ cm
$c = 6$ cm

$$s = \frac{a+b+c}{2}$$
$$= \frac{6+6+6}{2}$$

$\therefore$ semi-perimeter $= 9$ cm

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{9(9-6)(9-6)(9-6)}$$
$$= \sqrt{9 \times 3 \times 3 \times 3}$$
$$= 9\sqrt{3}$$

$\therefore$ area $\doteq 15.57$ cm^2

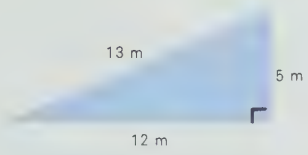
check
$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 6 \times 3\sqrt{3}$
$= 9\sqrt{3}$
$\therefore$ area $= 15.57$ cm^2

The next example was chosen so you can see that the "new" formula does agree with your other formula.

Example 2

Find the area of the right triangle shown.



Solution

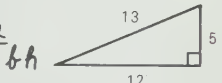
$$\begin{aligned} a &= 13 \text{ m} \\ b &= 12 \text{ m} \\ c &= 5 \text{ m} \\ a+b+c &= 30 \text{ m} \\ s &= \tfrac{30}{2} \\ \therefore \text{semi-perimeter} &= 15 \text{ m} \\ A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{15(15-13)(15-12)(15-5)} \\ &= \sqrt{15 \times 2 \times 3 \times 10} \\ &= \sqrt{5 \times 3 \times 2 \times 3 \times 5 \times 2} \\ &= 5 \times 3 \times 2 \\ \therefore \text{area} &= 30 \text{ m}^2 \end{aligned}$$

check

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 12 \times 5$

area $= 30 \text{ m}^2$



1. Calculate the value of the semi-perimeter "s" for triangles with the following lengths as sides.
   (*a*) 3 cm, 4 cm, 5 cm
   (*b*) 7 m, 6 m, 7 m
   (*c*) 4 km, 6 km, 8 km
   (*d*) 3 m, 5 m, 6 m

2. Calculate the area of each triangle having the given lengths as sides.
   (*a*) 3, 5, 6
   (*b*) 4, 3, 2
   (*c*) 14, 17, 13
   (*d*) 3, 4, 5
   (*e*) 8, 15, 17
   (*f*) 9, 12, 15
   (*g*) 2, 4, 5
   (*h*) 10, 24, 26
   (*i*) 5, 6, 7
   (*j*) 20, 25, 35
   (*k*) 25, 60, 65
   (*l*) 21, 28, 35

3. A triangular piece of tin with sides 45 cm, 36 cm, and 28 cm has a mass of 1.48 kg. Find the mass of a square metre of the material.

Maintaining Basic Skills 5

1. Using simple interest for your calculations, find the values needed to complete the table.

	Principal	*Rate of Interest*	*Time*	*Interest*
(*a*)	\$300	5%	3 years	
(*b*)		12%	7 years	\$210.00
(*c*)	\$800		5 months	\$20.00
(*d*)	\$600	8%		\$192.00
(*e*)	\$200	6%	4 months	
(*f*)		11%	9 months	\$33.00
(*g*)	\$100		219 days	\$5.40
(*h*)	\$50	14%		\$1.40
(*i*)		15%	292 days	\$18.00
(*j*)	\$80	12%	146 days	
(*k*)	\$1000		2 years	\$180.00
(*l*)	\$200	$5\frac{1}{2}$%		\$10.50

2. Using interest compounded as required, find for each part (*i*) an expression for the amount (*ii*) the actual amount using tables.

	Principal	*Rate*	*Time*	*Compounded*	*Amount (i)*	*Amount (ii)*
(*a*)	\$300	8%	6 years	annually		
(*b*)	\$60	10%	8 years	semi-annually		
(*c*)	\$500	12%	5 years	quarterly		
(*d*)	\$40	9%	15 years	annually		
(*e*)	\$200	7%	2 years	semi-annually		
(*f*)	\$130	24%	7 months	monthly		
(*g*)	\$2000	20%	5 years	quarterly		

3. Hy Silo borrows \$8000 to buy some new farm equipment. He agrees to pay it back in 4 years with 12% interest compounded semi-annually. At the end of the 4 years, how much does Hy owe?

4. On John's seventh birthday, his father invests a sum of money so that on his eighteenth birthday he will have \$5000 to go to college. The money was invested at 6% compounded annually. How much money did the father invest?

2(a) $300\,(1.08)^{6}$ \$476.06
(b) $60\,(1.05)^{16}$ \$130.97
(c) $500\,(1.03)^{20}$ \$903.06

3. \$12 750.80

4. \$2633.95

5. Mary and Barbara each earned money during the summer holidays. Mary invested \$160 at the end of July at 12% compounded monthly, and \$180 at the end of August at the same rate. Barbara invested \$340 at the end of September at 14% compounded semi-annually. At the end of June the next year, which girl had the larger amount, and by how much?

6. Copy and complete the following table.

Selling Price	*Prime Cost*	*Overhead % of Prime*	*Gross Cost*	*Markup % of Prime*	*Profit*
(*a*) \$310	\$200	25%	\$250	55% \$110	\$60
(*b*) \$600	\$400	20% \$80	\$480	50% \$200	\$120
(*c*) \$120	\$80	10%		50%	
(*d*)	\$60			\$40	\$30
(*e*)	\$500	\$50			\$200

7. Find the missing values in the table. Copy and complete.

Cost Price	*Selling Price*	*Profit*	*Rate of Profit*
\$370	\$450		
\$26	\$45		
\$50		\$30	
\$90			45%
\$6		\$6	
\$250			100%
	\$250	\$50	
	\$90		50%

8. Copy and complete the following table.

Cost Price	*Selling Price*	*Loss*	*Rate of Loss*
\$120			10%
\$380			5%
\$90	\$70		
\$150	\$100		
\$80		\$20	
	\$175	\$25	
\$12		\$4	
	\$800		20%

5. Mary had approximately 81¢ more than Barbara.

7.

Cost Price	*Rate of Profit*
\$370	21.6%
\$26	73.1%
\$50	60%

8.

Cost Price	*Loss*
\$120	\$12
\$380	\$19
\$90	\$20

9. The Decibel Shop buys a new stereo for $140. If a 60% profit is desired, what should the selling price be?

10. Al Varoom buys a new motor bike for $850. He rides it for 8 months then decides to sell it at a loss of 30%. What does Al charge for it?

11. A Christmas card company promises a profit of 40% to any organization selling their cards. The high school students want to earn $300 to buy new gym equipment. Cards cost the school 90 cents a dozen, and there is no overhead. How many dozen cards must the students sell?

12. Find the following amounts.

List Price	*Trade Discount*	*Amount Owing*
$300	25%	
$120	20%, 10%	
$50	$33\frac{1}{3}$%	
$80	30%, 5%	
$240	25%, 10%, 5%	

13. Find the amount owing to obtain the best discount.

Amount of Invoice	*Terms*	*Amount Owing*
$42	2/10, n/30	
$200	3/20, n/60	
$150	3/20, 2/30, n/40	
$1000	5/10, 1/20, n/30	
$120	4/10, 2/20, n/30	

14. Copy and complete the table.

Sales	*Rate*	*Commission*
$6500	3%	
$170	4%	
$12 600	0.5%	
	2%	$25
$320		$16

15. Jack Sellers, household appliances salesman, receives a salary of $100 per week plus a commission of 2% on sales up to $7000 and 1% on sales over that amount. In one week, he sold $9600 worth of washers and driers. How much did he make that week?

9. $224

10. $595

12.

List Price	*Amount Owing*
$300	$225.00
$120	$86.40

13.

Amount of Invoice	*Amount Owing*
$42	$41.16
$200	$194.00

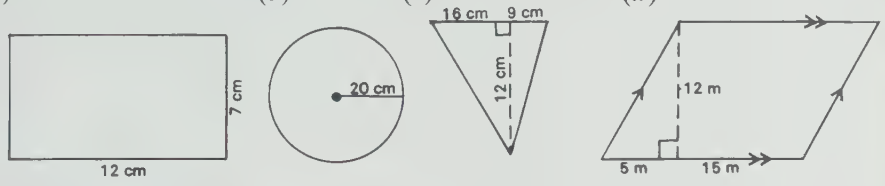
14.

Sales	*Commission*
$6500	$195.00
$170	$6.80

16. A real estate agent gets 5% commission on houses he sells, and 3% on land without a house. A house sold for \$66 000 and a small farm without buildings for \$90 000. What was the agent's commission?

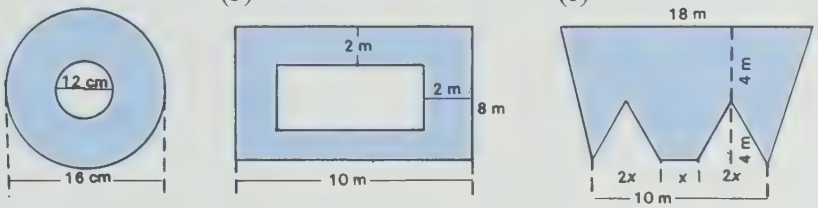
17. Find the perimeter and the area of each figure.

(*a*) (*b*) (*c*) (*d*)



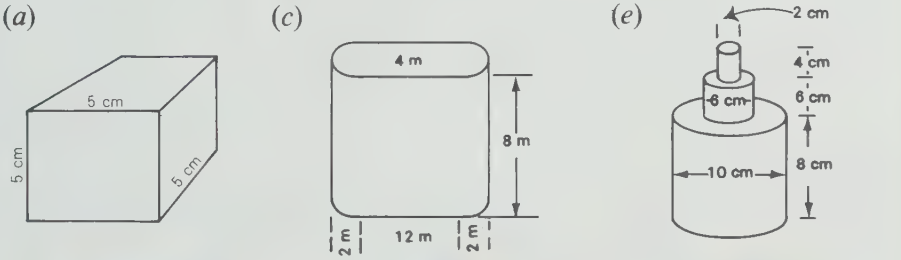
18. Find the area of the shaded section.

(*a*) (*b*) (*c*)

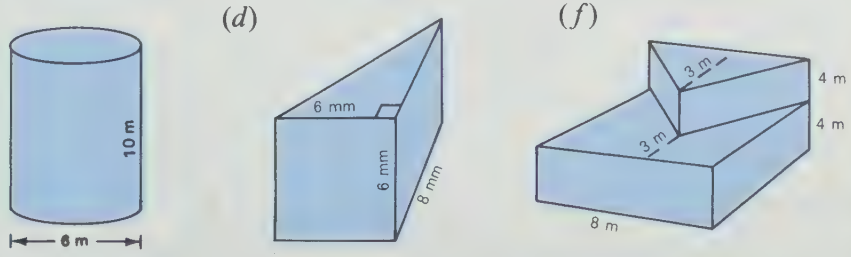


19. Find the surface area and the volume of each figure.

(*a*) (*c*) (*e*)



(*b*) (*d*) (*f*)



20. A cylindrical silo is 6 m high and has an inside radius of 5 m. The walls are 15 cm thick.
   (*a*) How much feed does the silo hold?
   (*b*) Find the area of the outside walls of the silo.
   (*c*) What is the cost of painting the outside, at \$1.10/m²?

16. \$6000

17(c) 60 cm, 150 cm²
(d) 66 m, 240 m²

18(a) 88 cm²

19(a) 150 cm², 125 cm³
(b) 245 m², 283 m³
(f) 280 m², 240 m³

20(c) \$213.46

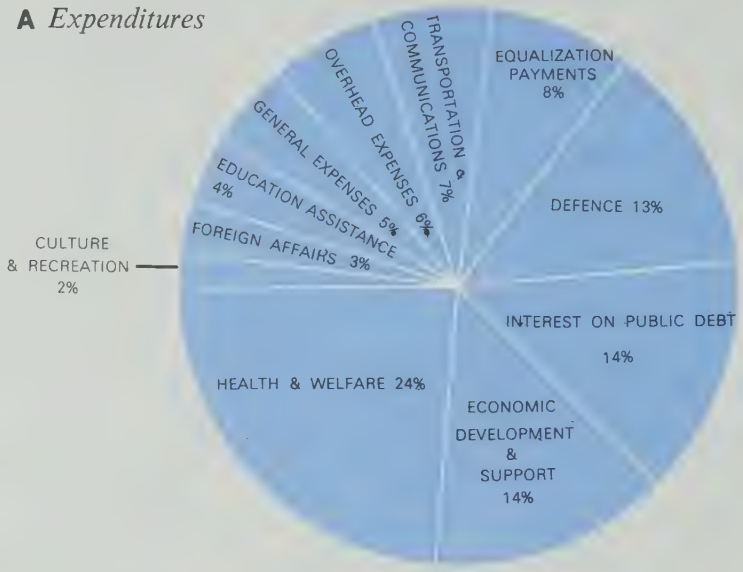
Unit 21

Taxes "The Cost of Being Canadian"

Canadians generally must contribute to the support of the activities of the three levels of government that run their public affairs. The three levels of government — federal, provincial, and municipal — together spend approximately one-half of the national income of the entire country. We will examine briefly the expenditures and the sources of revenue for each of the levels of government in this unit.

21.1 Federal Government

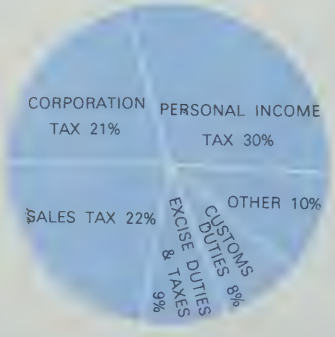
A *Expenditures*



The circle graph gives an approximate idea of how the money collected is spent. The actual figures vary from year to year depending upon the priorities established by the government in power in Ottawa.

B *Sources of Federal Revenue*

Under the British North America Act, the Parliament of Canada has the right to raise "money by any mode or system of taxation."



The main source of revenue for the Federal Government as shown in the circle graph is the tax placed upon the income of people employed in Canada. This is also the one tax of which every employed person is well aware. However, the graph will indicate that there are many other sources of income for the Federal Treasury.

Exercise 21.1

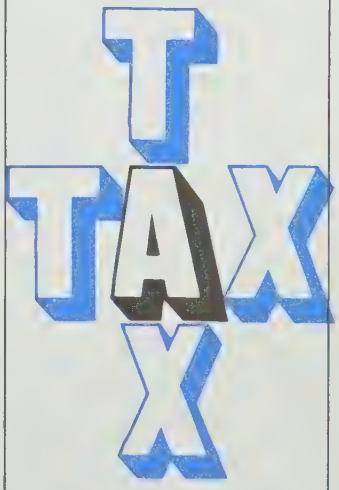
1. The total of all expenditures of the Federal Government is $19 billion.
   (*a*) Calculate the total spent on each category shown on the circle graph of expenditures.
   (*b*) Compare the amount spent on recreation by the government to the amount spent by the average taxpayer.
   (*c*) Compare the amount spent on transportation by the government to the amount spent by the average taxpayer.
   (*d*) Compare the amount spent on interest on debt by the government to the amount spent by the average taxpayer.

2. In the calculation of personal income tax, an individual is allowed to subtract amounts of money from his or her gross income before calculating the amount of tax to be paid. These amounts of money are called *personal exemptions*. Besides the basic personal exemption, a number of exemptions available to taxpayers are the married exemption, exemption for dependent children and other dependants, age and blind exemption, employment expense deduction, charitable donations, pensions and retirement savings plans, unemployment insurance, and medical expenses. Obtain the latest income tax form and the booklet supplied with it, and examine each of these exemptions in detail. You will need the information in order to be able to proceed in this exercise.

3. Determine the total personal exemptions for a young single healthy taxpayer living alone and having no dependants. You should ignore pension and unemployment insurance contributions for now and any unusual medical expenses.

4. Determine the married exemption for the following taxpayers.
   (*a*) Spouse earns $1250.00.
   (*b*) Spouse earns $200.00.
   (*c*) Spouse earns $1075.00.

5. What is the maximum that a spouse can earn before the maried exemption becomes zero?

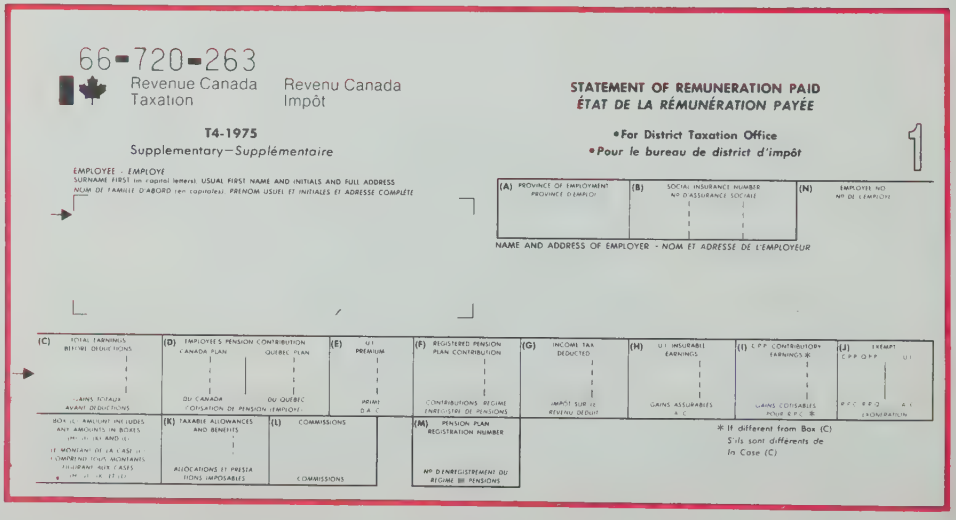


6. Determine the deductions for the following wholly-dependent children.
   (*a*) Boy age 17 earns $950.00.
   (*b*) Girl age 14 earns $680.00.
   (*c*) Girl age 18 earns $1275.00.
   (*d*) Boy age 15 earns $1500.00.

7. What is the maximum that a wholly-dependent child can earn before the exemption becomes zero if the child is
   (*a*) 16 or over,
   (*b*) under 16?

8. Discuss the possible reasons for the various exemptions allowed and their amounts.

Personal income taxes were introduced in 1917 as a temporary source of funds. Under the laws of Canada, an employer is now obliged to deduct income tax from every employee. An employer that either does not make the required deductions or makes the deductions but does not remit the amounts to the government is liable for fines and /or imprisonment. An employer having made all the necessary deductions is required to send a summary of these deductions to the employee. This summary is made by using the T4 slip shown here.



66-720-263

Revenue Canada Taxation — Revenu Canada Impôt

T4-1975
Supplementary—*Supplémentaire*

STATEMENT OF REMUNERATION PAID
ÉTAT DE LA RÉMUNÉRATION PAYÉE

•For District Taxation Office
•*Pour le bureau de district d'impôt*

1

EMPLOYEE – *EMPLOYÉ*
SURNAME FIRST (in capital letters). USUAL FIRST NAME AND INITIALS AND FULL ADDRESS
NOM DE FAMILLE D'ABORD (en capitales). PRÉNOM USUEL ET INITIALES ET ADRESSE COMPLÈTE

(A) PROVINCE OF EMPLOYMENT / PROVINCE D'EMPLOI
(B) SOCIAL INSURANCE NUMBER / N° D'ASSURANCE SOCIALE
(N) EMPLOYEE NO. / N° DE L'EMPLOYÉ

NAME AND ADDRESS OF EMPLOYER – *NOM ET ADRESSE DE L'EMPLOYEUR*

(C) TOTAL EARNINGS BEFORE DEDUCTIONS / *GAINS TOTAUX AVANT DÉDUCTIONS*
(D) EMPLOYEE'S PENSION CONTRIBUTION — CANADA PLAN / QUEBEC PLAN — *DU CANADA / DU QUÉBEC — COTISATION DE PENSION-EMPLOYÉ*
(E) U.I. PREMIUM / *PRIME D'A.-C.*
(F) REGISTERED PENSION PLAN CONTRIBUTION / *CONTRIBUTIONS RÉGIME ENREGISTRÉ DE PENSIONS*
(G) INCOME TAX DEDUCTED / *IMPÔT SUR LE REVENU DÉDUIT*
(H) U.I. INSURABLE EARNINGS / *GAINS ASSURABLES A.-C.*
(I) C.P.P. CONTRIBUTORY EARNINGS* / *GAINS COTISABLES RPC/RPQ**
(J) EXEMPT C.P.P. Q.P.P. U.I. / *EXONÉRATION RPC RPQ A.-C.*

BOX (C) AMOUNT INCLUDES ANY AMOUNTS IN BOXES (D), (E) AND (I)
LE MONTANT DE LA CASE (C) COMPREND TOUS MONTANTS FIGURANT AUX CASES (D), (E) ET (I)

(K) TAXABLE ALLOWANCES AND BENEFITS / *ALLOCATIONS ET PRESTATIONS IMPOSABLES*
(L) COMMISSIONS / *COMMISSIONS*
(M) PENSION PLAN REGISTRATION NUMBER / *N° D'ENREGISTREMENT DU RÉGIME DE PENSIONS*

* If different from Box (C)
S'ils sont différents de la Case (C)

The taxpayer on receipt of this slip is able to complete his income tax form.

9. I. M. Broke is a married man with two children under 16 years old. His T4 slip shows his total earnings for the year to be $8700.00, his C. P. P. contributions $88.20 and his income tax deductions $1183.00. His wife and children had no income for the year and he had medical expenses of $143.00, charitable donations of $50.00 and union dues of $72.00. Prepare the necessary income taxes for Mr. Broke. Will he have to pay more or will he get a refund?

You will need the most recent income tax forms for these questions.

10. U. R. Rich, a married man with 3 children, two of which are over 16 years old, had total annual earnings of $18 800.00 paid $4448.40 in income taxes, $88.20 to the C. P. P. and $1069.80 to a registered pension fund and had the following expenses. Prepare his tax form.
    (*a*) Medical expenses $235.00
    (*b*) Charitable donations $350.00
    (*c*) Professional dues $150.00

11. Y. B. Puzzled has a problem. His 18-year-old son has a summer job which pays him $105.00 per week and he has almost completed his tenth week. If he works more than 10 weeks his father will begin to lose him as a deduction. What they are trying to decide is whether it would be worthwhile for the boy to work the three remaining weeks of his holidays or whether he should take a vacation. Mr. Puzzled estimates that his personal income for the year will be $14 400.00 on which he will have paid $3082.20 tax, $88.20 C. P. P. and $631.80 to a registered pension fund. (Mrs. Puzzled has no income.) What would you recommend?

12. Complete the required income tax form for each of the situations outlined below.

    (*a*) Married man, wife's income $840, son age 19 earned $1200, daughter age 14 earned $900, son age 9 no income, supports widowed mother.

Income	$9600.00
Income tax deducted	$1116.00
C. P. P.	$ 88.20
Registered pension fund	$ 768.00
Professional fees	$ 75.00
Medical expenses	$ 295.00
Charitable donations	$ 150.00
Interest on bonds	$ 60.00

(*b*) Single girl, 19 years old, living with her parents, and going to school has a part-time job.

Income	$2300.00
Income tax deducted	$ 241.00
C. P. P.	$ 35.38

(*c*) Assume the girl in (*b*) obtained a good permanent job and left home.

Income	$8500.00
Income tax deducted	$1474.20
C. P. P.	$ 120.60

Discuss the two situations outlined in (*b*) and (*c*).

(*d*) Married man, supports wife and five very young children.

Income	$16 200.00
Income tax	$ 2 840.70
C. P. P.	$ 120.60
Medical expenses	$ 580.00

(*e*) Married man whose wife works and earns $8500 has a 19-year-old daughter going to school who earned $1350 and a 17-year-old son also at school who earned $900.

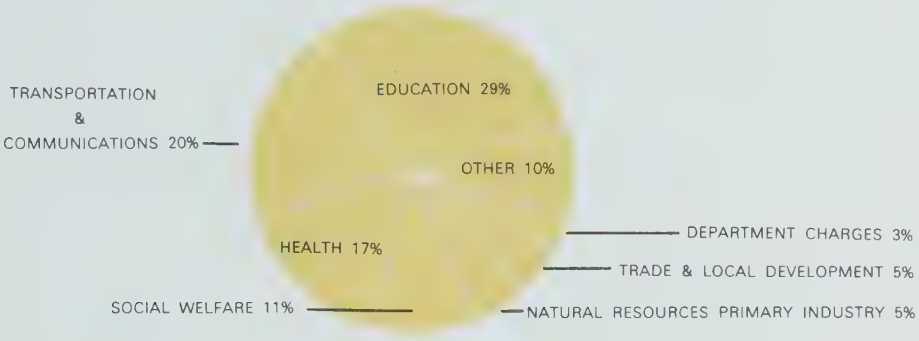
Income	$24 000.00
Income tax	$ 7481.40
Interest on bonds	$ 1800.00
C. P. P.	$ 120.60
Registered retirement fund	$ 2400.00
Professional dues	$ 250.00
Medical expenses	$ 350.00
Charitable donations	$ 750.00

13. Corporation taxes, sales taxes, excise duties and taxes, and customs duties constitute the other main sources of revenue for the Federal Treasury.
    (*a*) How is each collected?
    (*b*) How much money is collected from each?
    (*c*) From whom is this money collected?
    You will have to do some research in order to complete this question. The latest *Canada Year Book*, available from the Queen's Printer, Ottawa is a valuable source of information.

21.2 Provincial Governments

A *Expenditures*

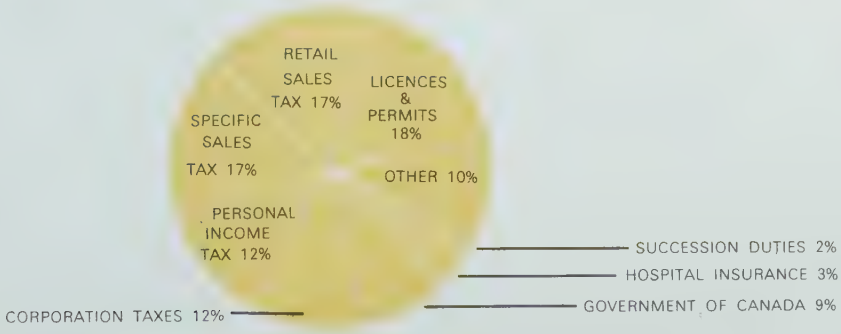
Each provincial government requires funds in order to carry on its duties. The circle graph shows how the provincial governments together spend their money.



1. Compare these various sources shown to the sources of the Federal Government.
2. Who is providing the money in each case?

B *Sources of Provincial Revenue*

Under the terms of the BNA Act, each provincial government is restricted to the use of *direct taxation* within the province in order to raise revenue. Direct taxes are those demanded from the very person who it is intended or desired should pay it. Thus the provinces are limited to the imposition of income tax, retail sales tax, succession duties, and other direct levies. The circle graph shows the sources for all provincial governments.



1. Discuss the expenditures of the provincial governments.
2. Compare these expenditures to those of the Federal Government. Do you see the main differences in responsibility?

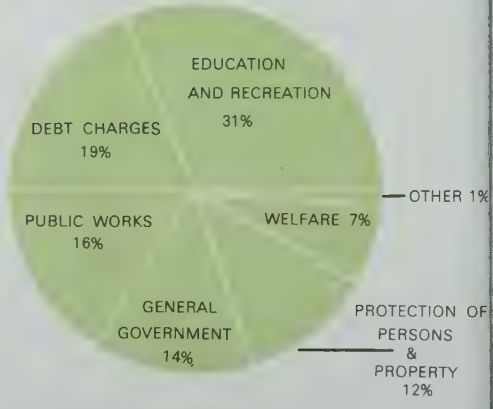
Research Exercise 21.2

1. Determine the actual expenditures of your Provincial Government. Make a circle graph showing these expenditures as percentages.
2. How do the expenditures of your Provincial Government compare
   (*a*) to another provincial government near you,
   (*b*) to all other provincial governments?
3. Determine the actual revenues of your Provincial Government. Make a circle graph showing these revenues as percentages.
4. Compare your Provincial Government's revenues to those of a neighboring province.
5. Can you think of any other direct sources of revenue for a provincial government to explore?
6. Do some research into the main sources of revenue. Who pays each? How is each collected?
   (*a*) licences and permits
   (*b*) retail sales tax
   (*c*) specific sales tax
   (*d*) personal income tax
   (*e*) corporation taxes
   (*f*) Government of Canada

21.3 Municipal Governments

A *Expenditures*

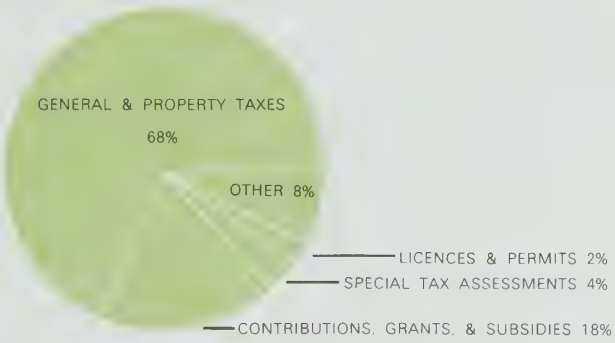
The expenditures of municipal governments are concentrated on activities within the particular municipality. The circle graph shows how municipal governments spend their money. Of course, individual municipalities will vary from this, but generally this is how the money is spent.



1. Discuss the expenses of the municipal governments. Do you see the main difference between the local, provincial, and federal expenditures?

Sources of Municipal Revenue

B Municipalities derive their powers from the provincial government concerned. Thus, municipalities are restricted to direct taxation sources in order to raise funds. The graph shows how municipalities generally raise money.



1. Discuss the revenues of municipal governments. Can you see any advantages and /or disadvantages to these sources? Can you suggest alternate sources? What are their advantages? What are their disadvantages?
2. Who is supplying the money for each source shown on the graph?

Research Exercise 21.3

1. Determine the actual expenditures of your municipality. Make a circle graph of these expenditures and compare your graph to that of the total of all municipalities shown in Section 21.3A.
2. Discuss the debt charges of each level of government.
3. Discuss the education expenses of each level of government.
4. Determine the actual revenues of your municipality. Make a circle graph of these revenues and compare your graph to that shown in Section 21.3B.
5. Do some research into the main sources of revenue. Who pays each? How is each collected?
   (*a*) property taxes
   (*b*) contributions, grants, and subsidies
   (*c*) special tax assessments
   (*d*) licences and permits

Unit 22
Ratio and Proportion

22.1 Ratio and Proportion Revisited

A two-term ratio is defined as a pair of numbers used to compare two similar quantities.

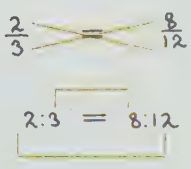
A Get-Set Exercise

1. Express the following ratios in lowest terms.
   (*a*) 4:12 (*d*) 3:27 (*g*) 16:12
   (*b*) 5:10 (*e*) 9:3 (*h*) 21:14
   (*c*) 6:9 (*f*) 25:15 (*i*) 8:2

2. State which of the following are equivalent ratios.
   (*a*) 7:14, 9:18 (*f*) 15:30, 18:36
   (*b*) 5:15, 3:9 (*g*) 1.5:2, 6:8
   (*c*) 8:18, 12:30 (*h*) 2:3.5, 10:14
   (*d*) 15:10, 8:12 (*i*) 19:3, 57:3
   (*e*) 36:9, 4:16 (*j*) 17.4:27, 52.2:81

3. State how you would test whether two ratios are equivalent.

REMEMBER?

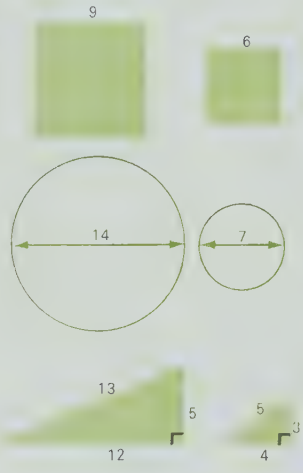


4. Use the test for equivalent ratios to find the value of each variable.

(*a*) $6:5 = 12:x$	(*f*) $33:3 = x:6$
(*b*) $2:3 = 8:x$	(*g*) $45:18 = 30:x$
(*c*) $18:12 = x:3$	(*h*) $x:34 = 3:2$
(*d*) $15:x = 3:4$	(*i*) $5:x = 65:26$
(*e*) $x:18 = 9:2$	(*j*) $2:x = x:32$

5. State the ratio of the following (in lowest terms).

(*a*) 12 cm to 6 m	(*c*) 50 m to 1 km
(*b*) 2 h to 15 min	(*d*) 400 g to 3 kg



6. For the two squares find the ratio of
   (*a*) the sides,
   (*b*) the perimeters,
   (*c*) the areas.

7. For the two circles shown, find the ratio of
   (*a*) the diameters,
   (*b*) the radii,
   (*c*) the circumferences,
   (*d*) the areas.

8. Find the ratios of the quantities indicated for the two triangles.
   (*a*) the perimeters
   (*b*) the areas
   (*c*) the altitudes

B *Proportion*

An equation involving two ratios is a *proportion*.

$$\frac{4}{5} = \frac{12}{15}$$

First proportional ——▼ or ——▼ Fourth proportional

$$4:5 = 12:15$$

Second proportional ——▲ ——▲ Third proportional

If the second and third proportional are equal, as in

$$\frac{2}{8} = \frac{8}{32}$$

then 8 is called a *mean proportional* between 2 and 32.

1. Find the missing proportional.
   (a) $\frac{3}{6} = \frac{x}{8}$
   (b) $\frac{15}{5} = \frac{9}{x}$
   (c) $\frac{x}{2} = \frac{27}{18}$
   (d) $\frac{15}{x} = \frac{30}{8}$
   (e) $\frac{23}{17} = \frac{x}{34}$
   (f) $\frac{x}{6} = \frac{15}{9}$
   (g) $\frac{12}{x} = \frac{27}{36}$
   (h) $\frac{x}{49} = \frac{18}{42}$

2. Find the first proportional in each case.
   (a) 6, 45, 54
   (b) 14, 6, 4
   (c) 21, 4, 7
   (d) 81, 9, 27

3. Calculate the following.
   (a) the third proportional to 3, 16, and 18
   (b) the fourth proportional to 3, 5, and 9
   (c) a mean proportional between 2 and 8
   (d) the second proportional to 4, 12, and 15
   (e) a mean proportional between 4 and 16
   (f) the fourth proportional to 3, 32, and 9
   (g) the first proportional to 3, 8, and 12
   (h) a mean proportional between 5 and 20

If $\frac{3}{6} = \frac{x}{8}$
then $6 \times x = 3 \times 8$
and $6x = 24$

Hint 2 (a)
$\frac{x}{6} = \frac{45}{54}$

Hint 3 (c)
If a mean proportional is x,
then $\frac{2}{x} = \frac{x}{8}$

Exercise 22.1

Written Solutions

1. A classroom measures 14 m $\times$ 10 m. What is the ratio of the length to the width?
2. If a school football team played 10 games and won 7 of them, what is the ratio of losses to wins?
3. If you earned \$12 480 per year and had to pay \$260 per month for rent, what fraction of your income would this represent? Does that seem reasonable?
4. If there are 380 students in grade 11 and 76 of them are on the honor roll, what fraction would this represent?
5. A family covered 2900 km in the first five days of a camping trip. At this rate, how many days would it take to complete the trip if 8700 km remained?

1. 7:5
3. $\frac{1}{4}$
5. 15 days

Would you like to take this trip?

6. If the water supply to your backyard pool can supply water at a rate of 575 ℓ in 3 min, how long would it take at this rate to fill a pool that holds 46 kl?

7. If the Capable Carwash Carousel can handle 1 car every 4 min, how many could it handle in 12 h?

8. A jetliner can make a 900 km trip in 1 h and 15 min. At the same rate, how long would it take to go 2100 km?

9. Radio signals travel at the speed of 299 790 km/s (approximately). If Astro Nott is standing on the moon, approximately 386 000 km away, what would be the minimum time lag between the end of his question and the beginning of an answer from earth as heard by Astro?

10. A riveted joint 100 cm long requires 25 rivets. With the same spacing, how many rivets would be required for a 6 m length?

11. A bar of square steel 66 cm in length has a mass of 8.8 kg. What would be the mass of a similar piece of steel 145 cm long?

12. The Kwik Konstruction crew of 55 men can build 10 houses in 6 weeks. How many men would be required to construct 75 houses at the same rate?

13. If a nuclear-powered submarine can circumnavigate the globe in 44 days, how long would it take 4 such submarines?

7. 180 cars
9. 2.6 s
11. 19.3 kg

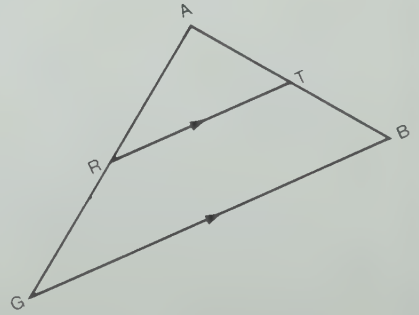
22.2 Proportions in Geometry

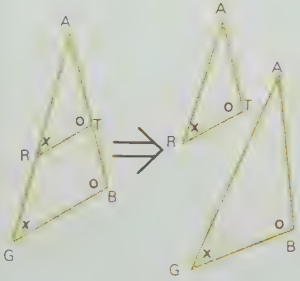
A *Similar Triangles*

In the $\triangle AGB$ a line RT has been drawn so that RT$||$GB. From your knowledge of parallel lines you will realize that

$\angle ART = \angle AGB$ (F pattern)
$\angle ATR = \angle ABG$ (F pattern)

This means that $\triangle ART$ and $\triangle AGB$ have all three angles respectively equal.





"Respectively equal" means every angle in $\triangle ART$ has an equal angle in $\triangle AGB$.

Triangles such as these are called *similar* triangles. It is interesting to compare the lengths of the sides of similar triangles. Measurement of the sides of $\triangle ART$ and $\triangle AGB$ will show

$$AR = 2 \text{ cm} \qquad AG = 4 \text{ cm}$$
$$AT = 1.5 \text{ cm} \qquad AB = 3 \text{ cm}$$
$$RT = 2.5 \text{ cm} \qquad GB = 5 \text{ cm}$$

From these measurements, calculate the value of each of the following.

$$AR{:}AG \quad AT{:}AB \quad RT{:}GB$$

What do you observe about these three ratios?

Actually, it can be proved that, for any pair of similar triangles ART, AGB,

$$\frac{AR}{AG} = \frac{AT}{AB} = \frac{RT}{GB}$$

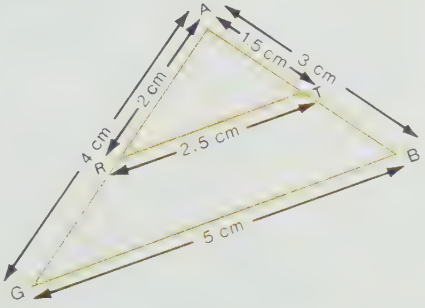
$\triangle ART$
$\sim \triangle AGB$

Corresponding sides are opposite the equal angles.

B *Division of Segments By a Parallel Line*

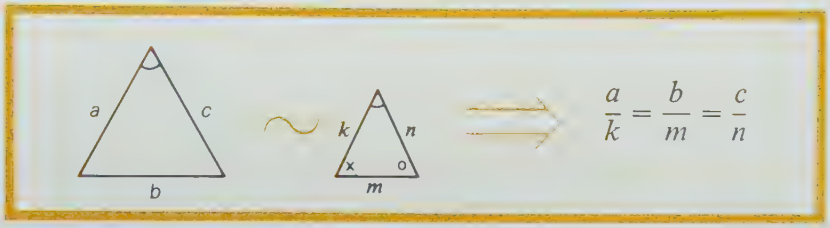
With a ruler and pencil draw any triangle and mark a point anywhere on one of the sides. Draw a line through this point parallel to one of the sides and extend the line to meet the other side of the triangle. Are the triangles similar? Why? By measurement, check whether the ratio of the corresponding sides is equal. Can you find any other ratios that are equal?

Completing the drawing as described, we have added the lengths as measured. Simple subtraction would reveal that RG = 2 cm and TB = 1.5 cm.

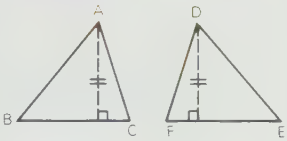


$$\frac{AR}{RG} = \frac{2 \text{ cm}}{2 \text{ cm}} = \frac{1}{1} = \frac{1.5 \text{ cm}}{1.5 \text{ cm}} = \frac{AT}{TB}$$

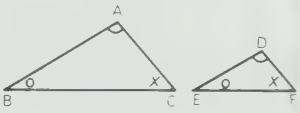
Consequently it appears as if the parallel segment RT forms four segments AR, RG, AT, and TB that are in proportion.



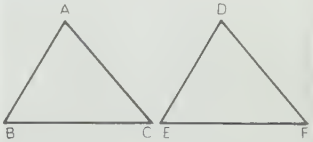
Three symbols you should know



$\triangle ABC = \triangle DEF$
(equal in area)
same size



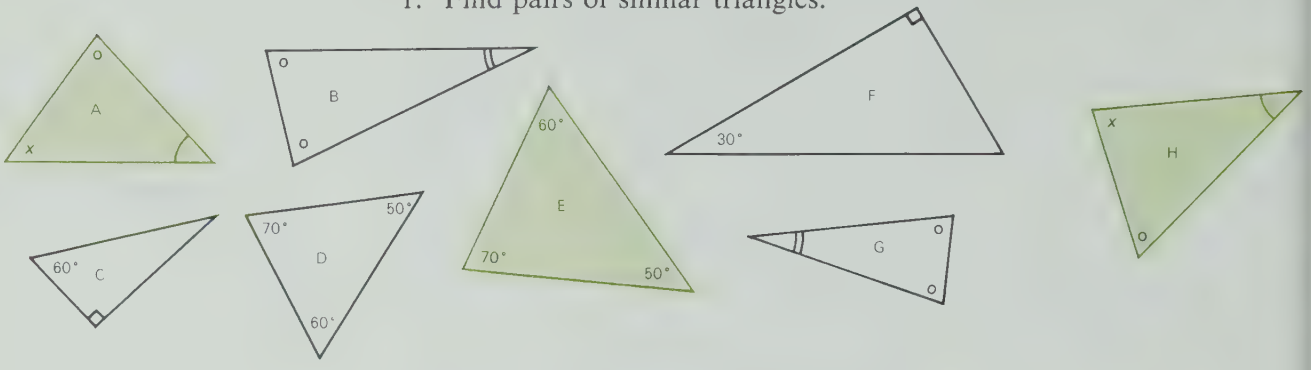
$\triangle ABC \sim \triangle DEF$
(similar)
same shape



$\triangle ABC \cong \triangle DEF$
(congruent)
same size *and*
same shape

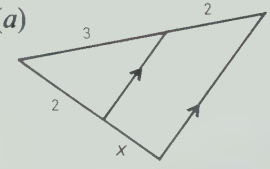
Exercise 22.2

1. Find pairs of similar triangles.

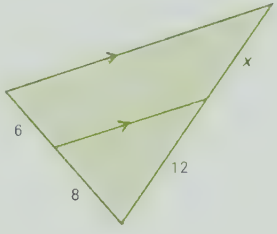


2. Calculate the missing dimensions.

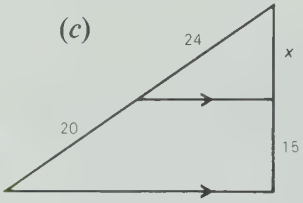
(*a*)



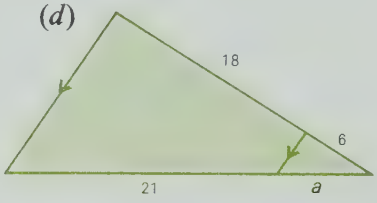
(*b*)



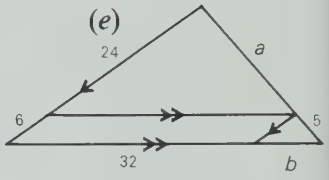
(*c*)



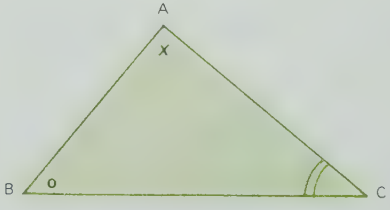
(*d*)

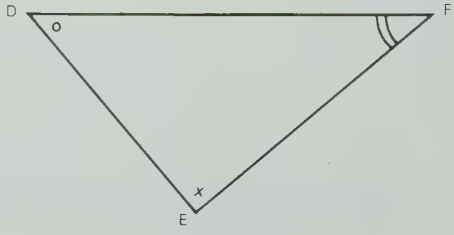


(*e*)



3. $\triangle ABC$ is similar to $\triangle DEF$.





$\triangle ABC \sim \triangle ???$

Find the sides of $\triangle DEF$ corresponding to the following sides of $\triangle ABC$.

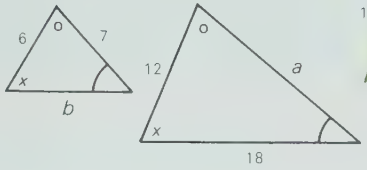
(*a*) AB (*b*) BC (*c*) CA

4. Use the diagram in Question 3 to write three ratios that are equal.

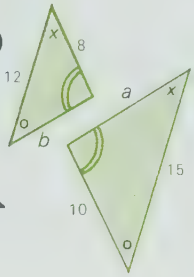
1. A ~ H
   B ~ G
2. (a) $\frac{4}{3}$
   (d) 7
3. (a) ED
   (b) DF
   (c) FE

5. Calculate the missing lengths for each pair of similar triangles.

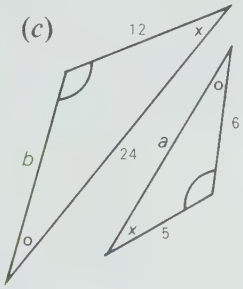
(a)



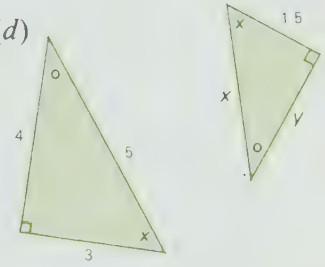
(b)



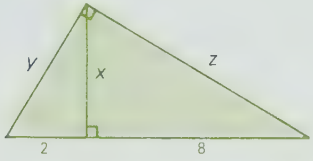
(c)



(d)



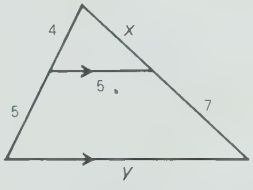
6. Here's a tough one. Find x, y, and z.



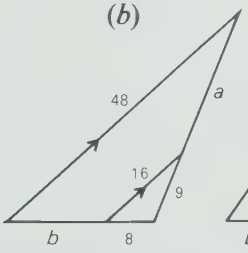
Hint: Look for three similar triangles.

7. Calculate the missing lengths indicated.

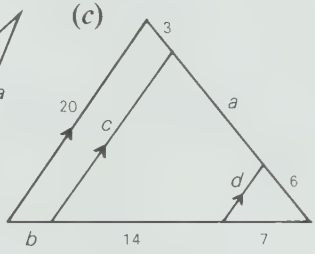
(a)



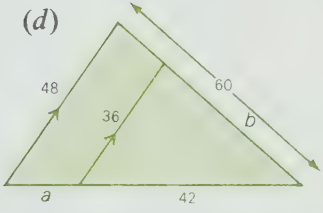
(b)



(c)

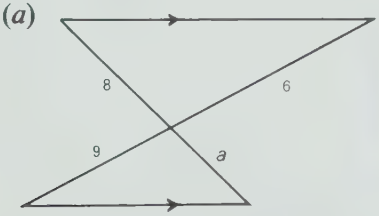


(d)

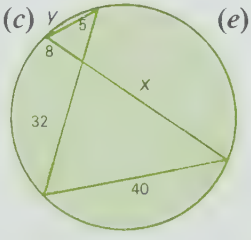


8. Calculate the missing lengths.

(a)



(c)



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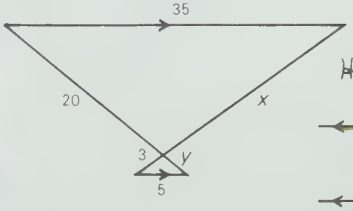
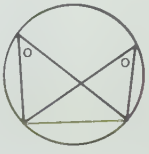


Hint #1

Hint #2



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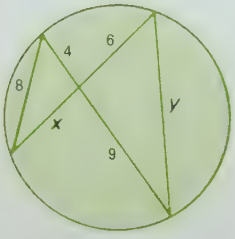

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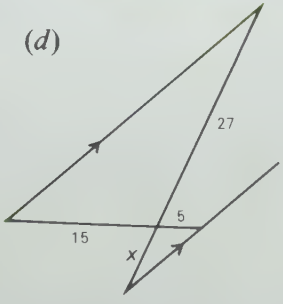

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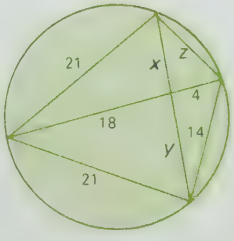
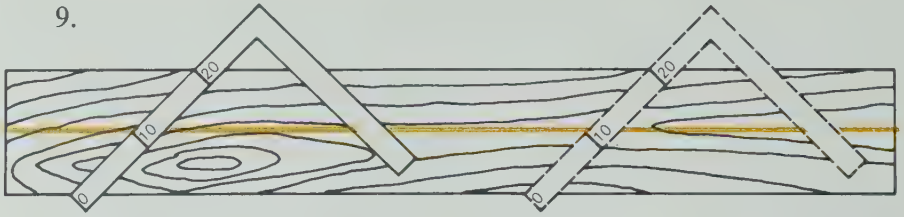


5.(a) $a = 14$, $b = 9$
(b) $a = 10$, $b = 8$

7.(a) $x = 5\frac{3}{5}$, $y = 11\frac{1}{4}$
(b) $a = 18$, $b = 16$

8.(a) $a = 12$
(b) $x = 6$, $y = 12$
(c) $x = 20$, $y = 10$

9.



This is how a carpenter would use his square to draw a line down the centre of a board. How could he divide the board lengthwise into

(*a*) four sections?

(*b*) three sections?

10. How could you use the lines on a ruled page to bisect a line segment 5 cm long? How could you trisect the same segment using lined paper?

22.3 Gear Drives

A *Gear Drives*

The proportion $\frac{T_d}{T_f} = \frac{N_f}{N_d}$ is used in calculations involving gear drives.

T_d Number of teeth in driver.

T_f Number of teeth in follower

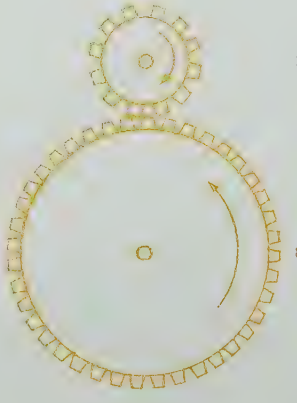
N_d Speed of driver in revolutions per minute.

N_f Speed of follower in revolutions per minute.

Example 1

In a gear train, the driver gear has 24 teeth and turns at 120 r/min. What is the number of revolutions per minute of the follower if it has 80 teeth?

Solution



$$\frac{T_d}{T_f} = \frac{N_f}{N_d}$$

$$\therefore \frac{24}{80} = \frac{x}{120}$$

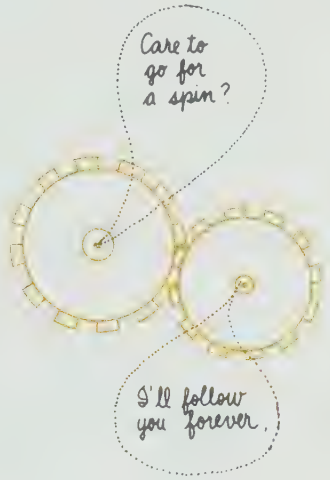
$$\therefore 80x = 120 \times 24$$

$$x = \frac{120 \times 24}{80}$$

$$= 36$$

$\therefore$ the follower turns at 36 r/min.

When two gears mesh, the direction of rotation of the driver is always opposite to that of the follower. In Example 1, if the driver rotated clockwise, the follower would rotate counter-clockwise. If it is desired to have the final gear rotate in the same direction as the driver, an *idler gear* is inserted. This is called a *simple train.*



Example 2

An idler gear having 40 teeth is inserted between the two gears in the previous example. Determine the speed of rotation of the idler and follower.

Solution

Using $\frac{T_d}{T_f} = \frac{N_f}{N_d}$ where the idler is considered as the follower

$$\frac{24}{40} = \frac{x}{120}$$

$$\therefore 40x = 24 \times 120$$

$$x = \frac{24 \times 120}{40}$$

$$= 72$$

$\therefore$ the idler turns at 72 r/min.

Now using $\frac{T_d}{T_f} = \frac{N_f}{N_d}$ where the idler is considered as the driver

$$\frac{40}{80} = \frac{y}{72}$$

$$80y = 40 \times 72$$

$$y = \frac{40 \times 72}{80}$$

$$= 36$$

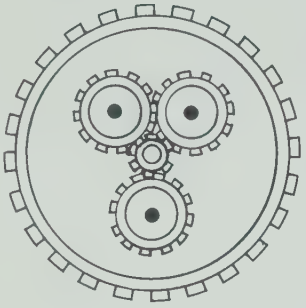
$\therefore$ the follower turns at 36 r/min.

This is the same answer as before, but now the direction is clockwise.

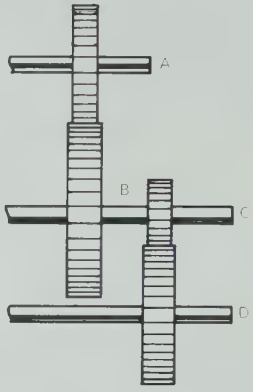
SIMPLE TRAIN

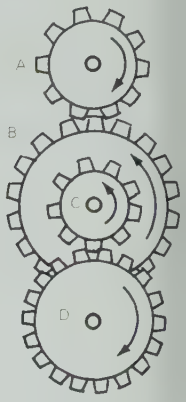
Does it matter how many teeth the idler has?

COMPOUND TRAIN



A *compound train* of gears is shown in the diagram. Note that gears B and C are both on the common shaft and so have the same direction of rotation.





Example 3

In the compound train pictured above, the driver A has 60 teeth and turns at 120 r/min (cw). If B, C, and D have 80, 15, and 45 teeth respectively, what is the number of revolutions per minute of D?

Solution

$$\text{Using } \frac{T_d}{T_f} = \frac{N_f}{N_d} \text{ on A and B}$$

$$\frac{60}{80} = \frac{x}{120}$$

$$80x = 120 \times 60$$

$$x = \frac{120 \times 60}{80}$$

$$= 90$$

$\therefore$ B turns at 90 r/min(ccw).

Because B and C are on the same shaft, C will also turn at 90 r/min.

$$\text{Using } \frac{T_d}{T_f} = \frac{N_f}{N_d} \text{ on C and D}$$

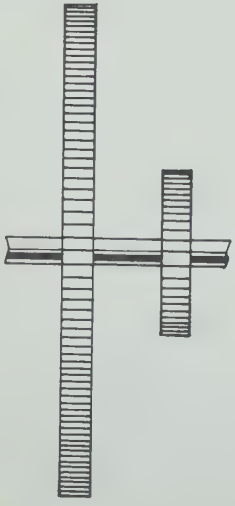
$$\frac{15}{45} = \frac{x}{90}$$

$$\therefore 45x = 90 \times 15$$

$$x = \frac{90 \times 15}{45}$$

$$= 30$$

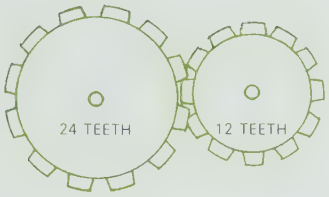
$\therefore$ D turns at 30 r/min(cw).



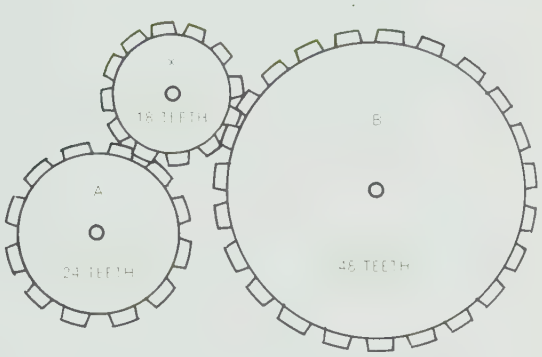
These two gears turn at the same number of revolutions per minute. Right?

Exercise 22.3

Discussion



1. (*a*) If the larger wheel is turned clockwise, in which direction will the smaller wheel turn?
   (*b*) How many turns does the small wheel make for the large wheel to turn around 3 times?
   (*c*) Which wheel will turn faster?

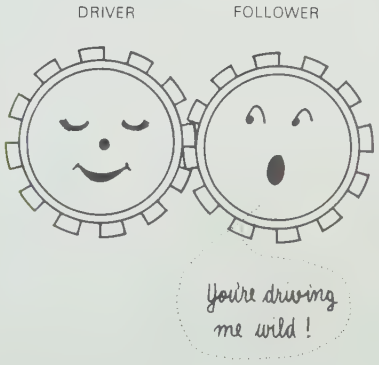


2. (*a*) If A is turned, will B turn in the same direction?
   (*b*) How many times does B turn when A turns once?
   (*c*) Repeat (*b*) with X having 36 teeth.

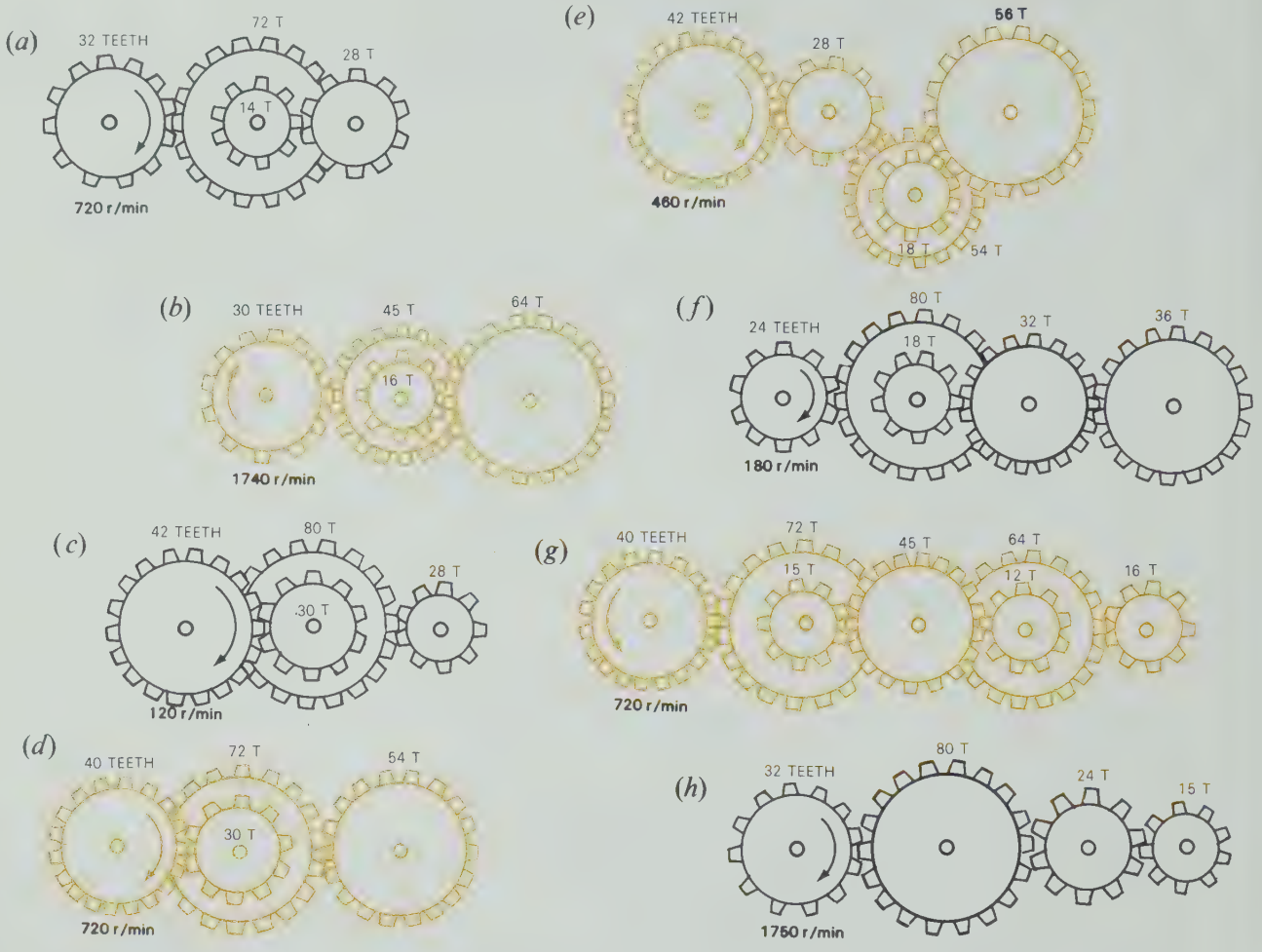
Written Solutions

3. Compute the missing values in the table for the simple trains described.

	DRIVER		FOLLOWER	
	Number of teeth	r/min	Number of teeth	r/min
(*a*)	36	100	?	150
(*b*)	?	300	96	200
(*c*)	33	80	24	?
(*d*)	24	160	?	60
(*e*)	28	320	64	?
(*f*)	24	?	40	135
(*g*)	80	?	75	160
(*h*)	80	50	?	200
(*i*)	24	1200	?	225
(*j*)	26	570	78	?
(*k*)	54	50	180	?
(*l*)	30	1700	24	?
(*m*)	42	?	30	140



4. Determine the speed and direction of the final gear in each compound train.



22.4 Belt Drives and Chain Drives

A *Belt Drives*

Belt drives have a wide application. The fan and generator of a car, and the fan in a hot-air furnace all use a belt drive.

The proportion $\frac{D_d}{D_f} = \frac{N_f}{N_d}$ is used in calculations involving belt drives.

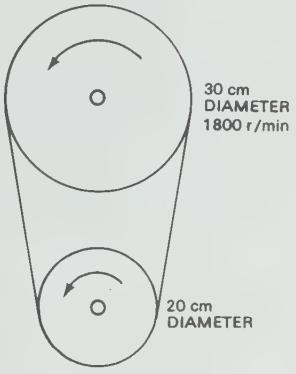
Here D represents the diameter of the pulley and N represents the number of revolutions per minute.

4.(a) 160 r/min, cw
(b) 290 r/min, cw
(e) 115 r/min, ccw
(f) 27 r/min, ccw

Example

A pulley on the shaft of a motor has a diameter of 30 cm. If the motor turns at 1800 r/min, how fast would another pulley 20 cm in diameter turn if they are connected by a belt?

Solution



$$\text{Using } \frac{D_d}{D_f} = \frac{N_f}{N_d}$$

$$\frac{30}{20} = \frac{x}{1800}$$

$$20x = 1800 \times 30$$

$$x = \frac{1800 \times 30}{20}$$

$$\therefore x = 2700$$

$\therefore$ the follower pulley rotates at 2700 r/min.

Cross – belt type. (Opposite direction of rotation.)

Chain Drives

Chain drives are generally used when the distance between shafts is too small to use belts and yet too great to use gears. The speeds involved are not as great as when using belt drives. The calculations, however, are identical to those of belt drive problems.

Example

The driver sprocket of a chain drive is 30 cm in diameter and rotates at 120 r/min. If the follower sprocket is to rotate at 80 r/min what diameter must it be?

Solution

$$\frac{D_d}{D_f} = \frac{N_f}{N_d}$$

$$\frac{30}{x} = \frac{80}{120}$$

$$80x = 120 \times 30$$

$$x = \frac{120 \times 30}{80}$$

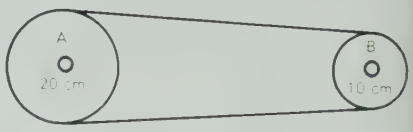
$$= 45$$

$\therefore$ the follower sprocket must be 45 cm in diameter.

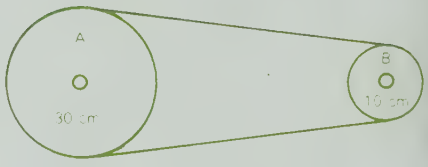
Exercise 22.4

Discussion

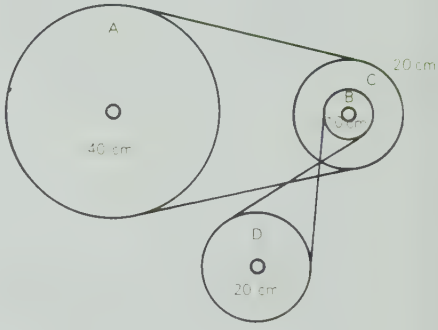
1. (*a*) If A is turned clockwise which way will B turn?
   (*b*) For each complete turn of A how many times will B turn?
   (*c*) Which pulley turns faster?



2. (*a*) If A is turned clockwise which way will B turn?
   (*b*) For each complete turn of A, how many turns will B make?
   (*c*) Which pulley turns faster?



3. (*a*) If A is turned clockwise, which way will D turn?
   (*b*) How many turns will D make for each turn of A?

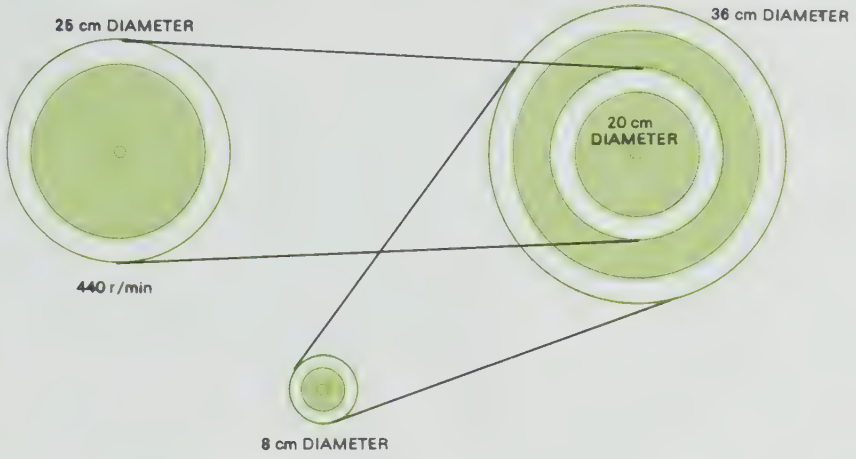


Written Solutions

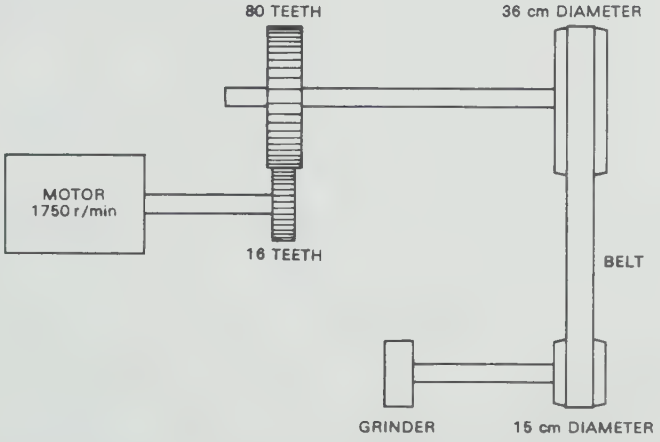
4. Compute the missing values in the table for the belt drives described.

	DRIVER		FOLLOWER	
	diameter (centimetres)	r/min	diameter (centimetres)	r/min
(*a*)	6	1500	18	?
(*b*)	6	400	?	300
(*c*)	12	?	8	800
(*d*)	14	1800	12	?
(*e*)	3	1600	4	?
(*f*)	9	?	6	900
(*g*)	20	?	8	2250
(*h*)	?	250	9	750
(*i*)	?	450	12	150
(*j*)	12	1200	?	800
(*k*)	18	400	?	200
(*l*)	12	?	9	1460

5. Calculate the speed of the shaft on which the 8 cm pulley is situated.



6. Calculate the speed of the grinder.



7. Walt Woodsplitter bought a new workshop power saw equipped with a 5 cm diameter pulley for belt drive. The instructions read, "Safe speed for this saw is 1400 r/min maximum." Walt has a motor that turns at 1750 r/min. What size of pulley should he buy for the motor?

8. The driver sprocket on Rusty Ironleg's bicycle has 96 teeth and rotates, when Rusty is in good shape, at 120 r/min. If the follower sprocket has 24 teeth, and the bicycle wheels are 66 cm in diameter, determine the speed of the bicycle.

5. 2475 r/min
6. 840 r/min
7. 4 cm

23

Unit 23
Mathematics and Chance

23.1 Probability

A Questions about probability usually occur whenever we play games with dice. When a die is rolled it is equally likely that any one of its six faces will land pointing upward. Since there is one way out of six for a die to turn up any one of its numbers, we say that the probability of rolling that number is $\frac{1}{6}$. For example, the probability of rolling a three is $\frac{1}{6}$.

If an experiment can result in N different, equally likely, outcomes and if S of the events correspond to a success, then we define the probability of success for this experiment by

$$P(\text{success}) = \frac{S}{N}$$

and the probability of failure by

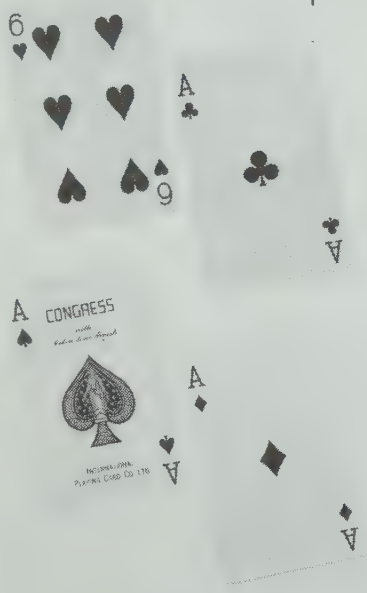
$$P(\text{failure}) = \frac{N - S}{N}.$$

Let's use this definition to figure out the probabiity of rolling an odd number with a die. Since the odd numbers on the face of the die are one, three, and five,
S, the number of successes, is 3, and
N, the number of different outcomes, is 6.

$$P(\text{odd number}) = \frac{S}{N}$$
$$= \frac{3}{6}$$
$$= \frac{1}{2}$$

What is the probability of not rolling an odd number?

$$P(\text{not an odd number}) = \frac{N - S}{N}$$
$$= \frac{3}{6}$$
$$= \frac{1}{2}$$



1. In a single throw of one die, what is the probability of success in throwing the following?
   (*a*) a one
   (*b*) a six
   (*c*) a seven

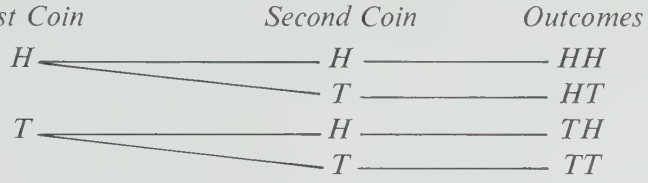
2. In drawing one card from a deck of cards, what is the probability of success in drawing the following?
   (*a*) a red ace
   (*b*) a red card
   (*c*) the Ace of Hearts
   (*d*) an ace

3. If a die is thrown, what is the probability of the number
   (*a*) failing to be a three?
   (*b*) being more than a three?
   (*c*) being an even number?
   (*d*) being an odd number?
   (*e*) failing to be a five?

4. (*a*) When a coin is tossed 999 times, what is the probability a head will show on the next toss?
   (*b*) What is the probability that the next child born will be a girl?

B If a coin is tossed 10 times, what do you think the chances are that it might come up heads each time? Suppose 10 coins are tossed at once. What is the probability that exactly half of them will come up heads and the other half tails?

A tree diagram helps to determine the set of outcomes of an event and from this diagram we can calculate the probability of an event. For example, let's construct a tree diagram and show all the possible outcomes of tossing two coins.

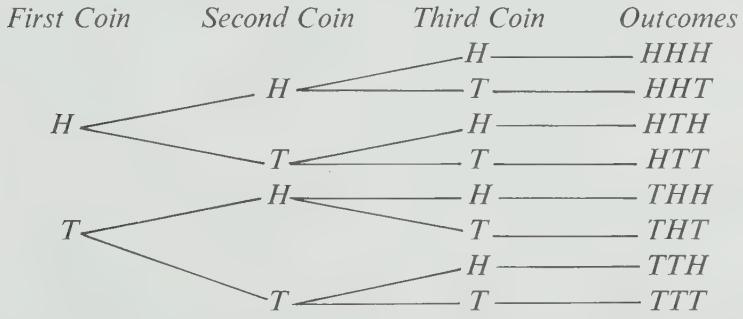


What is the probability of getting two heads (HH)? The total number of outcomes is four and only one of these is a success.
$\therefore S = 1, N = 4$
$\therefore P(HH) = \frac{1}{4}$.

What is the probability of obtaining two tails (TT)?

$P(TT) = \frac{1}{4}$

If three coins are tossed we extend the tree diagram as follows.



There are eight different equally likely outcomes. What is the probability of three heads? First find N and S.
$P(HHH) = \frac{1}{8}$ (probability of three heads)

What is the probability of two heads and one tail? You will notice in these outcomes that HHT, HTH, and THH all represent the event of two heads and a tail. Thus, $N = 8$ and $S = 3$.
$\therefore P$(two heads and a tail) $= \frac{3}{8}$.

Tree diagrams? Nothing to them!

1. Draw a tree diagram to show the possibilities of boys and girls in a family of two children.
   (*a*) In a family of two children, what is the probability that both are boys?
   (*b*) What is the probability that one is a boy and the other is a girl?

2. Draw a tree diagram to show the possibilities of boys and girls in a family of three children.
   (*a*) What is the probability that there is at least one girl in the family?
   (*b*) What is the probability of having all boys in the family?

3. Draw a tree diagram showing the outcomes when four coins are tossed.
   (*a*) What is $P(TTTT)$?
   (*b*) Calculate the probability of two heads and two tails.
   (*c*) Calculate the probability of three tails and a head.

C Probabilities can be used to find "odds." For example, a person might ask what are the odds in favor of rolling a five in one roll of a die. We know that $P(5) = \frac{1}{6}$, $P(\text{not a } 5) = \frac{5}{6}$. Thus, in six tries you would expect to win once and lose five times. We say that the odds are 1:5. Odds in favor can be calculated as

$$\frac{\text{Probability of success}}{\text{Probability of failure}}$$

$\therefore$ The odds of rolling a five in one roll of a die

$$= \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5} \text{ or } 1:5.$$

> The ratio of the probability that an event will occur to the probability that the same event will not occur is called the odds in favor of the event.

$$\frac{P(\text{success})}{P(\text{failure})} = \text{odds in favor}$$

1. On one roll of a die, what are the odds in favor of getting
   (*a*) a number less than five?
   (*b*) an odd number?

2. If the probability that an event will occur is $\frac{3}{5}$, what are the odds in favor of the event?

3. If the probability that a king or a six will be drawn from a deck of cards is $\frac{2}{13}$, what are the odds in favor of drawing a king or a six?

4. The odds of the Vancouver Canucks winning the series is 3:5.
   (*a*) In eight games, how many would they be expected to win?
   (*b*) If you bet three jellybeans in favor of Vancouver for one game, how many jellybeans would you expect to get if you won?
   (*c*) Show that betting on 3:5 odds, in the "long run" you would win no jellybeans.

Exercise 23.1

1. From a class of 40 students with 25 girls, what is the probability of choosing a boy?

2. A bag contains 10 white marbles and 8 red marbles. If one marble is drawn at random, what is the probability that it is
   (*a*) a white marble?
   (*b*) a red marble?

3. Louella Lastchance holds a lottery ticket. The lottery involves a total of 120 tickets, and 10 prizes are offered. What is the probability that Lou will *not* win a prize?

4. A letter is chosen at random from the name *Winnipeg*. What is the probability that the letter is
   (*a*) a vowel?
   (*b*) a consonant?
   (*c*) the letter *n*?

5. From a group of three students, a committee is to be chosen consisting of a president, a treasurer, and a secretary. By means of a tree diagram, illustrate the set of possible outcomes if the first choice will be president, the second choice treasurer, and the third choice secretary.

3. $\frac{11}{12}$

4.(a) $\frac{3}{8}$
(c) $\frac{1}{4}$

8. $\frac{13}{37}$

11.(a) $\frac{1}{1}$
(b) $\frac{1}{12}$

6. A student has in his pocket a penny, a nickel, a dime, and a quarter. He takes two coins out of his pocket, one after the other. Illustrate the possible outcomes with a tree graph. State the probability of choosing a nickel and a dime, not necessarily in order.

7. Draw a tree diagram to illustrate all the possible outcomes of a hockey series between two teams of equal ability that is to consist of three games. Find the probability of a team winning the series after it loses the first game.

8. The probability that an event will occur is 0.26. What are the odds that the event will occur?

9. A bag contains seven red balls and four black balls. If one ball is drawn, what are the odds in favor of its being a red ball? A black ball?

10. If a single die is thrown, what are the odds in favor of the number showing being greater than three?

11. A card is drawn at random from a deck of 52 playing cards. State the following odds.
   (*a*) that the card is red
   (*b*) that the card will be an ace
   (*c*) that a king is drawn

23.2 The Fundamental Counting Principle

We used a tree diagram to describe the different outcomes of tossing two or three coins. The number of outcomes can be easily determined without drawing the tree diagram.

A boy has three different colored shirts and four different ties. How many shirt-tie arrangements can he make?

With each shirt he can use one of four ties and thus make up four shirt-tie arrangements. Since there are three shirts, he can make up 3×4 shirt-tie arrangements.

1. A girl has six skirts and five blouses. How many shirt-blouse arrangements can she make?
2. Five boys and six girls arrive at the tennis court. How many different girl-boy couples can we make?
3. (*a*) A car maker has seven different models and eight different colors. How many choices does a buyer have?
   (*b*) The car maker adds three different engines to the options. How many choices does a buyer have now?

The problems we have just completed illustrate the *Fundamental Counting Principle* which is stated in words as follows.

> To find the number of arrangements that involve several choices in succession, multiply the number of ways each choice can be made.

Example 1

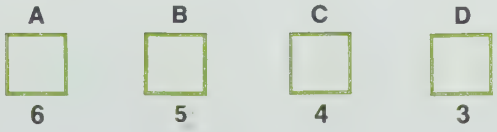
Suppose there are six vacant seats on a bus and four people enter. In how many ways can they be assigned seats?

Solution

Person *A* can be assigned a seat in six ways. Having filled one seat, then person *B* can be assigned a seat in five ways.

How many ways of assigning seats to persons *C* and *D* are there?

An easy way to show all this is to draw four boxes representing the people.



The numbers 6, 5, 4, and 3 indicate the number of ways each choice can be made. By the Fundamental Counting Principle, the total number of ways the people can be seated is

$$6 \times 5 \times 4 \times 3 = 360 \text{ ways}$$

Example 2

How many sets of answers could you have on a True or False test of five questions? (Assume each question is answered.)

Solution

Since five answers are requested, make five boxes in which each answer can be given.

Each question must be answered T or F, hence there are two ways of answering each question. Thus there are two ways of filling each box.

Therefore we could write

2	2	2	2	2

Now $2 \times 2 \times 2 \times 2 \times 2 = 32$.

$\therefore$ there are 32 possible sets of answers.

Exercise 23.2

1. A test of 10 questions can be answered true or false. How many different sets of answers could you have on the test?
2. In a Match-Mate Dating Service, if there are 40 men and 60 women, how many different dates can be arranged?
3. In a restaurant menu there are six different dinners, five different beverages, and seven different desserts.
   (*a*) If you order a dinner and a beverage how many different combinations can you choose?
   (*b*) If you just order a beverage and a dessert how many combinations are possible?
   (*c*) If you order a dinner, a beverage, and a dessert, what is the number of combinations you could choose?
4. In working out a solution to a crazy maze game there are six places where you could move to the left or the right as you go from the start to the finish. How many different paths are there to the maze game?

3.(a) 30
(c) 210
4. 64

5. A licence plate consists of two letters and three digits.
   (*a*) If any of the 26 letters of the alphabet can be used for each of the two letters, how many arrangements of letters are there?
   (*b*) If any of the 10 digits can be used for each of the three digits how many arrangements of digits are there?
   (*c*) How many different licence plates could you have?

6. The call letters of radio and television stations in Canada contain four letters and the first letter is always a C. How many sets of call letters are there?

7. Slot machines are made up of three wheels, each of which has 20 symbols consisting of cherries, oranges, plums, bells, bars, melons, and 7's. In how many ways can the symbols appear on a "pay line?"

8. The following lists the number of times the symbols are used on each wheel of a Las Vegas Machine.

	Wheel 1	*Wheel 2*	*Wheel 3*
Cherries	2	6	0
Oranges	5	5	5
Plums	7	3	4
Bells	1	5	9
Bars	3	2	1
Melons	2	2	3
7's	1	1	1

   (*a*) Show that two cherries (on wheel 1 and wheel 2) can come up on the pay line in exactly 240 ways.
   (*b*) Three melons is a jackpot. In how many ways can this jackpot come up?
   (*c*) What combination of three like symbols is the least likely to come up? This gives a double jackpot.

AM409

See "Complete Guide to Gambling" by Scarne.

7. 8000

8.(b) 12
(c) Three 7's

23.3 Permutations

In a recent art contest the judges had narrowed the contestants down until there were two finalists, *Untitled* and *Whimsy #2*. Clearly there were only two orders that could result: *Untitled* could be the winner and *Whimsy #2* the runner-up, or *Whimsy #2* could be the winner and *Untitled* the runner-up.

Suppose there were three finalists, *Untitled*, *Whimsy #2*, and *East of Halifax*. List the possible arrangements of the winner, the first runner-up, and the second runner-up.

Here are the six different ways in which the selections could be made.

Winner	*Untitled*	*U*	*U*	*W*	*W*	*EH*	*EH*
1st runner-up	*Whimsy #2*	*W*	*EH*	*U*	*EH*	*U*	*W*
2nd runner-up	*East of Halifax*	*EH*	*W*	*EH*	*U*	*W*	*U*

With four finalists the number of arrangements increases to 24. To illustrate, let A represent *Untitled*, B represent *Whimsy #2*, C represent *East of Halifax*, and D *Three Apples*.

Winner	A	A	A	A	A	A	B	B	B	B	B	B	C	C	C	C	C	C	D	D	D	D	D	D
1st runner-up	B	B	C	C	D	D	A	A	C	C	D	D	A	A	B	B	D	D	A	A	B	B	C	C
2nd runner-up	C	D	B	D	B	C	C	D	A	D	A	C	B	D	A	D	A	B	B	C	A	C	A	B
3rd runner-up	D	C	D	B	C	B	D	C	D	A	C	A	D	B	D	A	B	A	C	B	C	A	B	A

These arrangements are called *permutations*. A *permutation* is an arrangement of things in a definite order.

We can use the fundamental counting principle in the case of the four finalists. Each position can be represented by a box.

There are four ways of choosing the winner. Having chosen the winner, there are three ways left to choose the first runner-up, two ways to choose the second runner-up, and only one way to choose the third runner-up.

Thus, the number of ways of choosing the four finalists in a definite order is $4 \times 3 \times 2 \times 1 = 24$.

Similarly, the number of ways of choosing three finalists in a definite order is $3 \times 2 \times 1 = 6$, and for two finalists, $2 \times 1 = 2$.

A short way to write each of these products is to use a mathematical symbol that looks like an exclamation mark but is called a *factorial* symbol.

$4 \times 3 \times 2 \times 1$ can be written as $4!$
$3 \times 2 \times 1 = 3!$
$2 \times 1 = 2!$

In general, $n!$ means to multiply the consecutive whole numbers from n all the way down to 1, and the number of permutations of n different things is $n!$

Example

In how many ways can five people line up to buy their lunch?

Solution

We can represent the people with five boxes.

The first person in line could be any one of five, the second person any one of the four remaining, and so on.

We could show this by filling in the numbers in the boxes:

or by using the factorial notation 5!

$\therefore$ the number of ways five people could line up for lunch would be 120.

Exercise 23.3

1. In how many ways can seven different digits of a telephone number be arranged?
2. In how many ways can a hand of seven cards be arranged?
3. In how many ways can the names of five candidates for the same job be listed on a piece of paper?
4. In how many ways can eight people line up to buy tickets at a theatre box-office?

5. In how many ways can you arrange the letters of the word *meat*?

6. In how many ways can the first three positions be taken in a 100 m race of six athletes?

7. If there are 100 floats in the Rose Bowl Parade on New Year's Day, in how many different ways can the first and second prizes be chosen?

8. In a game of X's and O's, how many ways can the first three moves be made?

9. How many possible signals can be made using five different signal flags arranged one above the other?

10. In how many ways can eight boys be arranged in a row?

11. In how many ways can a president, secretary, and treasurer be chosen from a total of nine students?

12. In how many ways can the letters of the word *factor* be arranged?

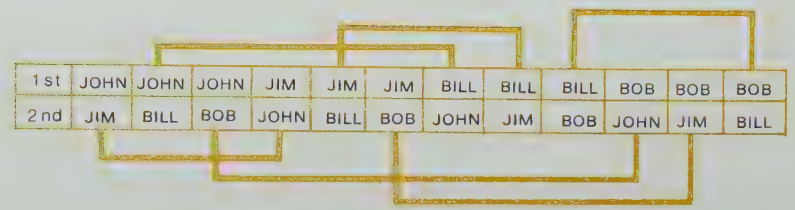
7. 9900
9. 120
11. 504

23.4 Combinations

There are seven candidates for the boys varsity basketball team. All of these boys want to be on the starting five. However the coach has told them that three players, Hank, Gord, and Harvey from last year's team, already have three positions made. That actually leaves four boys trying to fill two positions on the starting five.

Hank, Gord, and Harvey are already on the team so the remaining two are selected from John, Jim, Bill and Bob.

Using the fundamental counting principle, we could fill the first position four ways and the other position three ways, giving a total of 12 selections. But let's look at these selections.



1st	JOHN	JOHN	JOHN	JIM	JIM	JIM	BILL	BILL	BILL	BOB	BOB	BOB
2nd	JIM	BILL	BOB	JOHN	BILL	BOB	JOHN	JIM	BOB	JOHN	JIM	BILL

There are only six selections that result in a different team being formed. In the 12 selections, Jim-John and John-Jim are each counted—but they are not different selections for the team. Each pair is counted twice. Therefore the number of ways of selecting the starting team would be $\frac{4 \times 3}{2} = 6$.

This type of problem is called a *combination* problem. A *combination* is a selection of things in which the order does not matter.

Example

In a club with 10 members in how many ways can a committee of three be chosen?

Solution

By the fundamental counting principle, there would be $10 \times 9 \times 8$ ways to choose the committee in order. However, any three members can be arranged in 3! ways, without causing any change in committee membership.

$$\therefore \text{number of ways to choose the committee is: } \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$$

Exercise 23.4

1. There are five "stand-bys" waiting for seats on the "Fly-Nite Airways" flight to Thunder Bay. If there are three seats available on the plane, how many different combinations of "stand-bys" can be chosen?
2. A new Ontario government committee of five members is to be in charge of "Important Trivial Affairs." If only 100 members are willing to serve on the committee, how many different committees can be chosen?
3. Eight friends attend a reunion. Each shakes hands with each of the others. How many handshakes occur?
4. A new musical group is being formed called "Six Beats." It will consist of four guitar players and two drummers. At an audition 10 guitar players and five drummers try out.
   (*a*) How many combinations of guitar players can be chosen?
   (*b*) How many combinations of drummers can be chosen?

3. 28
4.(a) 210

8. 105
9. 2380

5. Find the number of combinations of the letters of the word *meat*, taking two at a time.
6. In how many ways can a team of five players be selected from 10 players?
7. In how many ways can a committee of three students be selected from seven students?
8. In a high school basketball league of 15 teams, each team plays every other team once. How many games must be scheduled?
9. In how many ways can a committee of four be selected from 17 different people?

Games People Play

Game 1 Roulette

One version of roulette is based on a round wheel with the numbers 1 to 36, 0 and 00 spaced evenly around the circumference and colored red or black. The wheel spins in one direction and a ball is rolled in the other. The bank wins if the ball comes to rest at 0 or 00 unless a player bets these numbers.

(*a*) If a player bets on a number, the bank pays off at 35 to 1. Assume that 36 players bet \$1 on each number.
  (*i*) Show that the bank loses nothing if one of the numbers wins.
  (*ii*) Show that the bank's odds of winning are $\frac{2}{38}$ or $5\frac{5}{19}\%$.
  (*iii*) Show that the bank wins approximately 26¢ on every \$5 bet.

(*b*) If a player bets red or black, the payoff is even money. (Remember the bank wins for 0 and 00.) Show that the percentage favoring the bank is $5\frac{5}{19}\%$.

(*c*) Describe the "Cancellation System" for playing roulette.

Game 2 Chuck-a-Luck

This game is played with three dice. Players bet on numbers 1 to 6. The players are paid the amount of their bet if their number turns up once, double if it turns up twice, and triple if their number turns up three times.

(*a*) In how many ways can three dice turn up?

(*b*) In a total of 216 plays, how many times would you expect to get a no-pair? A double? A triple?

(*c*) Assume that bettors cover each of the numbers with \$1. That is, \$6 is bet each time. For 216 games calculate the results for the following chart.

	House pays	*House takes*
no -pair		
double		
triple		

(*d*) In 216 games, $6 \times 216 = \$1296$ is bet. Show that the house's profit is \$102 and thereby show that the house's percentage win is $7\frac{47}{54}\%$.

Game 3 Dice Games

Two dice are rolled. In 36 rolls how often do you expect each number to turn up? Complete the chart in the margin.

(*a*) In the game of "Hardway" the numbers 4, 6, 8, or 10 have to be rolled with a pair. For example, if a player calls "4" and rolls a pair of two's, he wins. If he rolls 4 the easy way or rolls a 7 he loses.

(*i*) Find the correct odds for rolling a "hard 4."

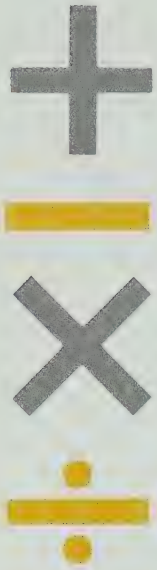
(*ii*) A casino offers 9 to 1 for a "hard 6" or a "hard 8." Show that the correct odds are 10 to 1.

(*b*) In "One Roll Action" bets are made on each roll of the dice. Here is a chart of the odds a casino offers. Calculate the correct odds.

The Bet	*Bank Pays*	*Correct Odds*
Two sixes (6 - 6) Two aces (1 - 1)	30 : 1	
Eleven (6 - 5) Three (1 - 2)	15 : 1	
All sevens (7)	4 : 1	
Any Crap (2, 3, or 12)	7 : 1	
7 with 3 - 4 7 with 5 - 2 7 with 6 - 1	15 : 1	

Number	Frequency out of 36
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	

Moral of the preceding questions: YOU CAN'T WIN



Maintaining Basic Skills 6

Use the following information for Questions 1 and 2.
Personal exemption $1878
Exemption for wife (earning not over $334) $1644
Federal Tax Rate:

Taxable Income	Tax
$3522	$ 616 + 21% on next $2348
5870	1109 + 23% on next 2348
8218	1649 + 25% on next 2348

Provincial Tax: 30.5% of the Federal Tax.

1. Mr. Greatly earned $13 280, spent $560 on a pension fund, and gave $285 to charity. His union fees were $70. He is married, with no children; wife's earnings $210. Calculate (*a*) his taxable income, (*b*) his income tax.

2. Miss Sprint, single, no dependants, finished school in June and earned $5760 for the remainder of the year as an office secretary. $1540 has been deducted from her pay check for income tax. Calculate her income tax. Does she pay more or get a refund?

3. Use the present income tax forms to calculate the income tax for the following people:
   (*a*) Mr. Black; single; salary $16 500; charity giving $185; pension plan $750.
   (*b*) Miss Brown; single; earned $8000; charity $370; pension $420; union dues $75.
   (*c*) Mr. White; married; earned $13 000; his wife earned $210; children age 2 and 4 years; charity $190; medical expenses $230; pension $900. Tax deducted at the source $1260.
   (*d*) Mr. Green; married; earned $19 000. Wife's earnings $205; children 6, 10, 15 years; charity $270; medical expenses $320; pension fund $560; tax paid $6000.
   (*e*) Mrs. Orange; widow; earned $12 420; she has a son 16 who earned $160; paid $540 for pension fund; already paid $2490 in tax. Does she pay more or get a rebate?

4. Find the value of the variables.
   (*a*) $3 : 5 = x : 20$
   (*b*) $x : 15 = 4 : 5$
   (*c*) $4 : q = 5 : 2$
   (*d*) $2 : y = 8 : 3$
   (*e*) $7 : 2 = 21 : t$
   (*f*) $3 : 7 = 4 : p$

4(a) 12
(b) 12
(c) $\frac{8}{5}$

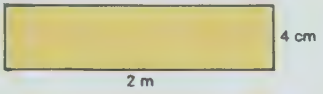
5. Solve for the variables.

(*a*) $\frac{1}{5} = \frac{2}{x}$ (*c*) $\frac{5}{3} = \frac{15}{r}$ (*e*) $\frac{t}{9} = \frac{1}{12}$

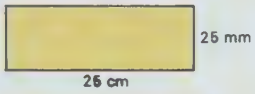
(*b*) $\frac{4}{7} = \frac{w}{14}$ (*d*) $\frac{3}{y} = \frac{12}{5}$ (*f*) $\frac{1}{s} = \frac{2}{5}$

6. Find the ratio of length to width.

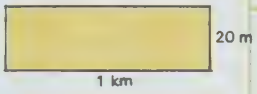
(*a*)



(*b*)

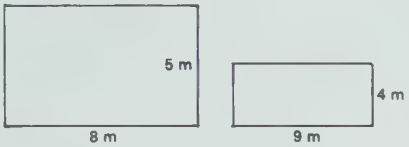


(*c*)

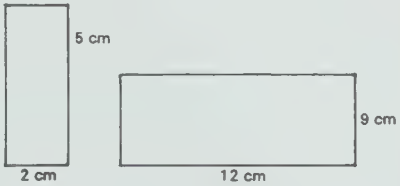


7. For each pair of rectangles, state the ratio of (*i*) the perimeters, (*ii*) the areas.

(*a*)

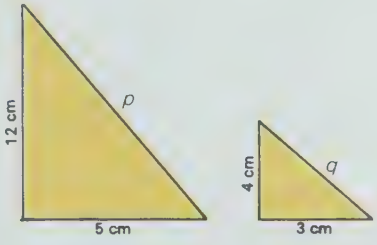


(*b*)

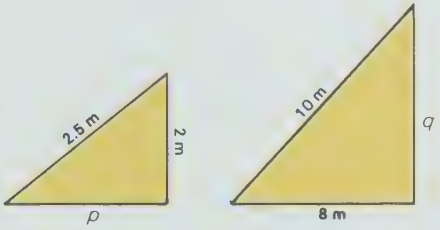


8. Find (*i*) the value of p and q, (*ii*) the ratio of the perimeters, (*iii*) the ratio of the areas of each pair of triangles.

(*a*)

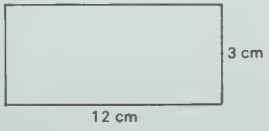


(*b*)

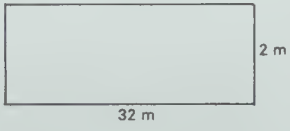


9. Find the side of a square that has the same area as the given rectangle.

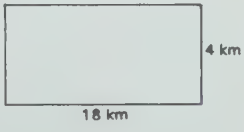
(*a*)



(*b*)



(*c*)



10. The mean proportional between 16 and m is 8. Find m.

5(a) 10
(b) 8
(c) 9

6(a) 50:1

7(a) (i) 1:1
(ii) 10:9

8(a) (i) $p = 13$ cm, $q = 5$ cm
(ii) 5:2
(iii) 5:1

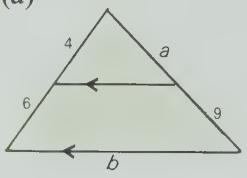
9(a) 6 cm

11. Calculate the missing values in the table for the simple gear trains described.

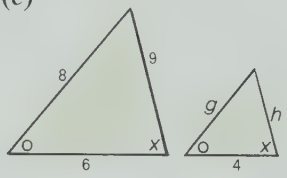
	Driver no. of teeth	r/min	Follower no. of teeth	r/min
(*a*)	24	200	40	
(*b*)	30	80		120
(*c*)	15	600	45	
(*d*)	49		35	140
(*e*)	60	360		120
(*f*)		17	21	68

12. Find the missing dimensions.

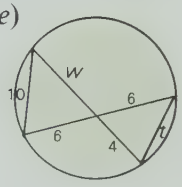
(*a*)



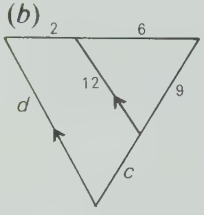
(*c*)



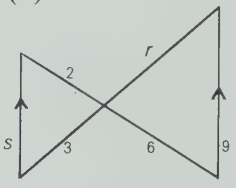
(*e*)



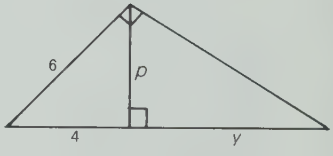
(*b*)



(*d*)



(*f*)



13. A quarter is tossed. Find the probability of (*a*) Head, (*b*) Tail, (*c*) Head or Tail.

14. One card is drawn from a deck of 52 cards. Find the probability the card is (*a*) red, (*b*) a queen, (*c*) a spade, (*d*) a jack of spades.

15. A red die and a green die are rolled. What is the probability that the sum surface is (*a*) three, (*b*) twelve, (*c*) seven, (*d*) eleven, (*e*) seven or eleven?

16. A bag of candies has 4 red, 5 green, and 2 white, all the same size. If you picked one with your eyes closed, what is the probability it would be (*a*) green, (*b*) red, (*c*) not white?

r/min

11(a) 120
(c) 200

12(a) $a = 6, b = 12$
(b) $c = 3, d = 16$
(d) $s = 3, r = 9$

13(a) $\frac{1}{2}$

14(a) $\frac{1}{2}$
(b) $\frac{1}{13}$

15(a) $\frac{1}{18}$
(b) $\frac{1}{36}$

16(a) $\frac{5}{11}$

17. A box has the following wooden letters: AAABCDDEFG. Dave is trying to find his initial, but the room is dark and he can't see. He picks a letter at random. What is the probability he gets the correct letter?

18. A test consists of four true or false questions. Jim forgot to study so he guessed at each one. What is the probability that he gets all four correct?

19. The probability that Jane passes an exam is $\frac{2}{5}$ (*a*) What are the odds in favor of her passing? (*b*) What is the probability that she fails? (*c*) With more studying, the odds in favor of passing are 7 : 3. What is the probability of passing?

20. Of 2000 tourists travelling the Banff-Jasper highway in one week, 300 saw a bear. What is the probability of one person seeing a bear?

21. The Weewin Insurance Company has insured each of 600 football players for $2000 against injury to a leg or arm that would prevent him from playing ball. The previous season it was found that out of 760 football players, 38 had hurt an arm or leg and couldn't play. The company wants to make $12 000 to include profit and expenses. How much premium should each of the players be charged for the insurance?

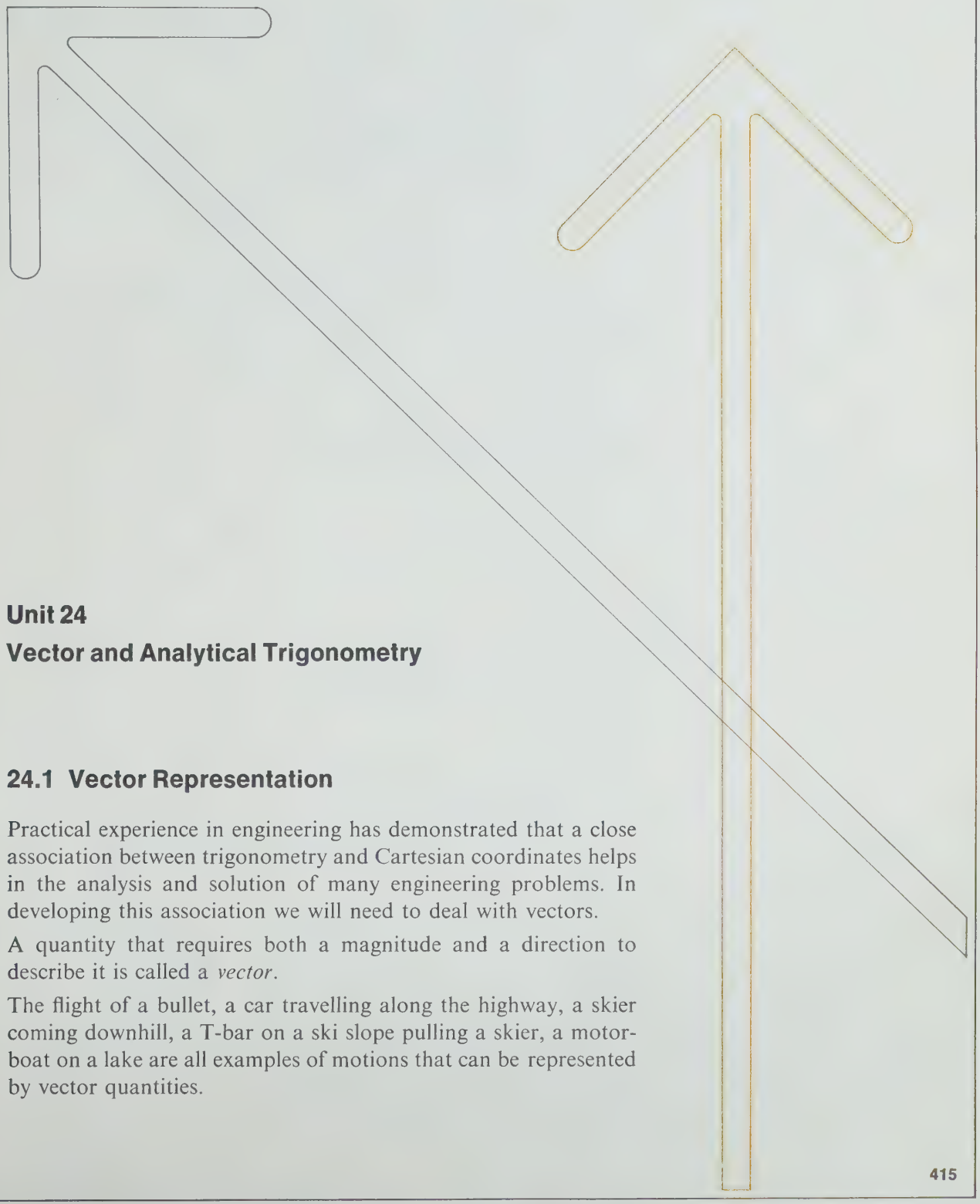
22. Of 300 eggs put into an incubator to hatch, 15 didn't hatch. (*a*) What is the probability of an egg hatching? (*b*) If a hatchery wanted 1200 chicks, how many eggs should be set?

17. $\frac{1}{5}$

19(a) $\frac{2}{3}$

(b) $\frac{3}{5}$

21. $120



Unit 24
Vector and Analytical Trigonometry

24.1 Vector Representation

Practical experience in engineering has demonstrated that a close association between trigonometry and Cartesian coordinates helps in the analysis and solution of many engineering problems. In developing this association we will need to deal with vectors.

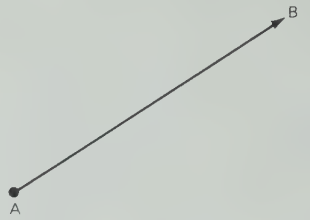
A A quantity that requires both a magnitude and a direction to describe it is called a *vector*.

The flight of a bullet, a car travelling along the highway, a skier coming downhill, a T-bar on a ski slope pulling a skier, a motorboat on a lake are all examples of motions that can be represented by vector quantities.

A directed line segment may be used to represent a vector quantity. The vector from A to B may be decribed in two ways.

(*a*) $\overrightarrow{AB}$ (read "vector AB") where A is the tail of the vector and B is the head which indicates the direction

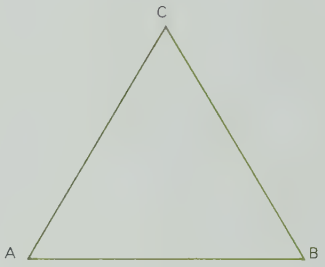
(*b*) $\vec{c}$ (read "vector *c*")



A *scalar* quantity involves magnitude or size only. Your age, the time of day, the number of students in your class are all scalar quantities.

Example 1

Using the line segments forming $\triangle ABC$, list all the possible vectors.

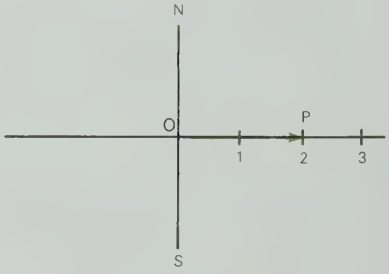


Solution

$\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$, $\overrightarrow{CA}$, $\overrightarrow{BC}$, $\overrightarrow{CB}$

Example 2

Represent a car travelling due east at 2 units per hour as a vector quantity.



Solution

$\overrightarrow{OP}$ represents a car travelling due east at 2 units per hour.

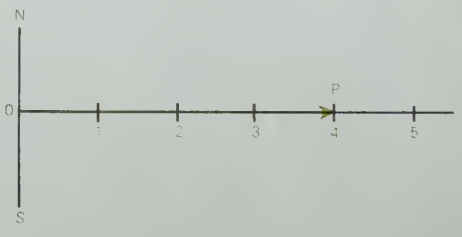
Scale diagrams help in showing the magnitudes of vectors.

Example 3

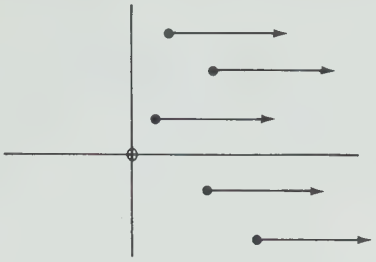
Represent a car travelling due east at 60 km/h as a vector quantity.

Solution

Let 1 cm represent 15 km/h. $\overrightarrow{OP}$ represents a car travelling east at 60 km/h and is therefore 4 cm long.



B Equal vectors have the same magnitude and direction. They do *not* have to begin at the same starting point.



All the vectors shown in the diagram at the left represent cars moving due east at 60 km/h. We call these equal vectors. They are equal in length and are parallel.

Example

From the following diagram select pairs of equal vectors. RSTV and PSTQ are parallelograms.

Solution

$\vec{RV} = \vec{ST}$ $\qquad$ $\vec{PQ} = \vec{ST}$

$\vec{RS} = \vec{VT}$ $\qquad$ $\vec{PS} = \vec{QT}$

$\vec{RV} = \vec{PQ}$

Line segment RS may be equal in length to line segment PS but $\vec{RS} \neq \vec{PS}$. Why?

C Opposite vectors have the same magnitude but are opposite in direction.

$\vec{XY}$ and $\vec{AB}$ are opposite vectors.

$\therefore \vec{XY} = -\vec{AB}$

or $\vec{AB} = -\vec{XY}$

We could also say

$\vec{XY} = -\vec{YX}$

and $\vec{AB} = -\vec{BA}$.

Notice that we can name any given vector in two ways using the end points.

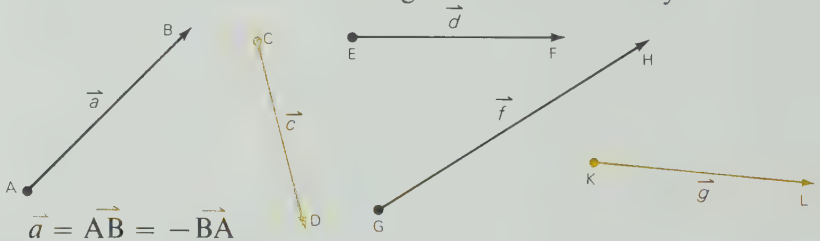
$\therefore \vec{PQ} = -\vec{QP}$

Exercise 24.1

Discussion

1. Which of the following are scalar quantities and which are vector quantities?
   (*a*) the motion of a puck travelling along the ice
   (*b*) the area of a triangle
   (*c*) your height
   (*d*) your mass
   (*e*) your age
   (*f*) the speed of a car
   (*g*) a force pulling a toboggan
   (*h*) the mass of a stone
   (*i*) acceleration due to gravity
   (*j*) the motion of a ball being hit by a baseball bat
   (*k*) the motion of a crane lifting a load from a truck

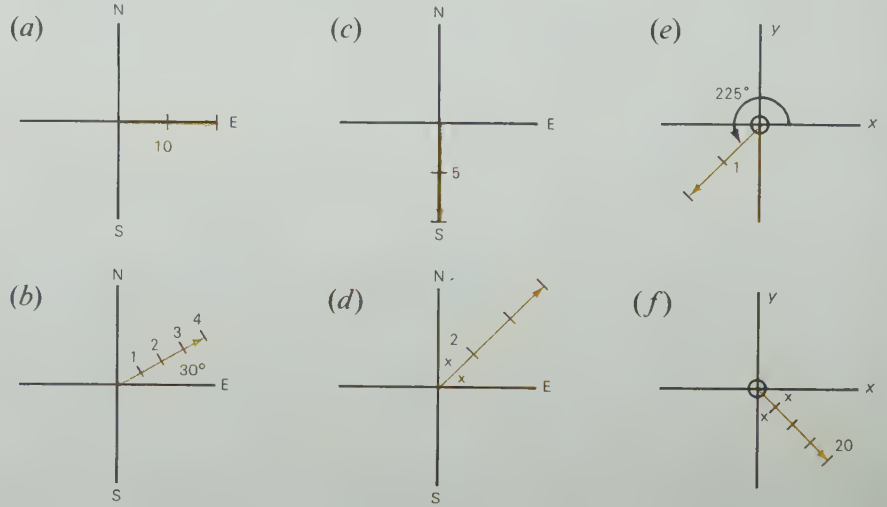
2. Name each of the following vectors in three ways.



$\vec{a} = \overrightarrow{AB} = -\overrightarrow{BA}$

Written Solutions

3. Describe in words the magnitude and direction of each of the following vectors.

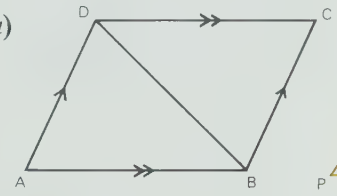


3.(a) 20 units, East
(c) 10 units, South
(f) 20 units, 315°

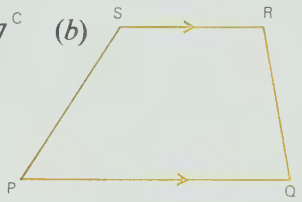
4. Use a suitable scale to draw vectors representing each of the following.
   (*a*) a sailboat travelling due north at 15 km/h
   (*b*) a wind from the northwest at 20 km/h
   (*c*) a force of 100 N acting straight up
   (*d*) a rocket travelling upwards at 500 m/s
   (*e*) acceleration due to gravity of 9.8 m/s^2
   (*f*) a plane taking off at 300 km/h at an angle of lift-off of 30° to the ground
   (*g*) a trip of 6 km due west
   (*h*) a skier travelling 100 m down a hill with a slope of $\frac{1}{2}$

5. List all the vectors that can be found in the following diagrams.

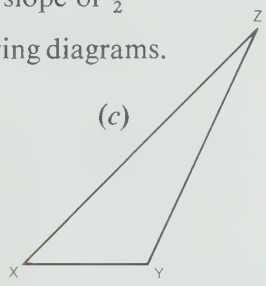
(*a*)



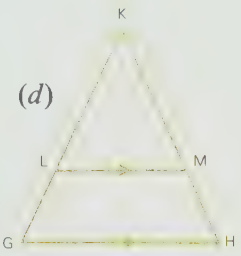
(*b*)



(*c*)



(*d*)



6. Ray Gatta can row a boat in still water at 8 km/h.
   (*a*) Represent this velocity with a vector if the direction he rows is due north.
   (*b*) If the water current is 3 km/h due south represent this quantity by a vector.
   (*c*) With what velocity relative to the shore would Ray row if he rowed due south? Represent this quantity with a vector diagram.

7. Consider the situation in which three towns are joined by highways as shown.

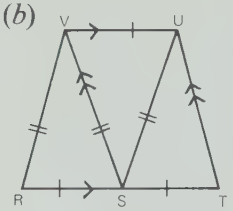
   (*a*) Name all possible vectors that represent trips between the three towns.
   (*b*) Choose a suitable scale. Draw a scale diagram.

5.(b) $\vec{SR}$, $\vec{PQ}$
(c) None
7.(a) $\vec{AB}$, $\vec{BA}$, $\vec{BC}$, $\vec{CB}$, $\vec{AC}$, $\vec{CA}$

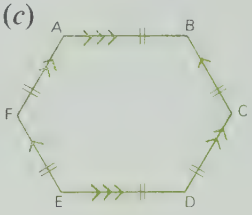
8. From the following diagrams select pairs of equal vectors and pairs of opposite vectors.

(a)

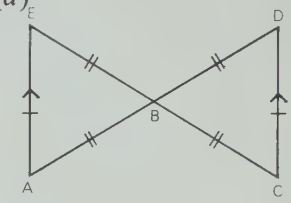
(b)



(c)

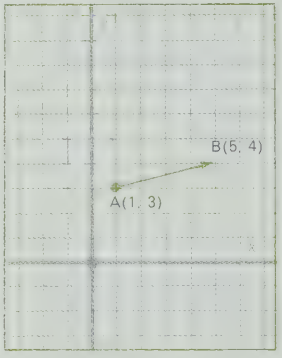


(d)

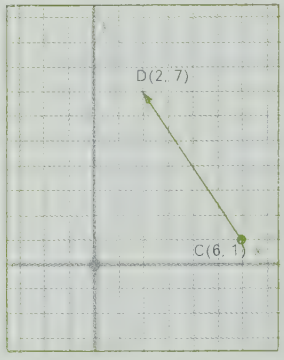


24.2 Vectors as Ordered Pairs

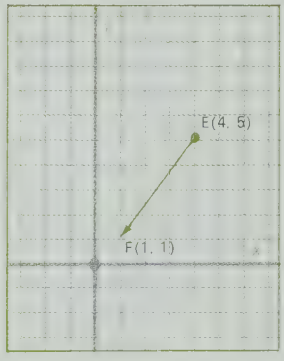
A As long as our vectors all lie in one plane, this plane can be regarded as the Cartesian plane, and each vector can be described using ordered pairs.



$\overrightarrow{AB}$ is the vector from (1, 3) to (5, 4)

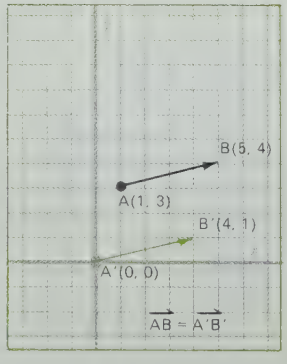


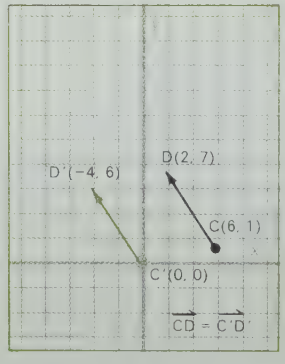
$\overrightarrow{CD}$ is the vector from (6, 1) to (2, 7)

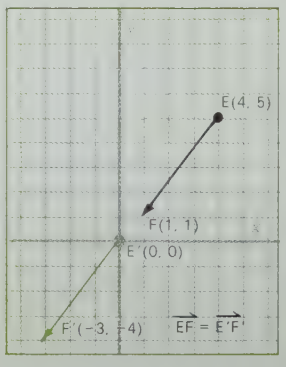


$\overrightarrow{EF}$ is the vector from (4, 5) to (1, 1)

Now let's draw equal vectors for each of the above with initial point at (0, 0).

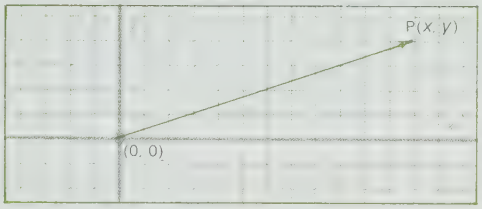






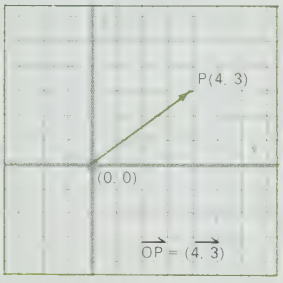
From this we deduce that any vector can be equated to an equal vector whose starting point is (0, 0)

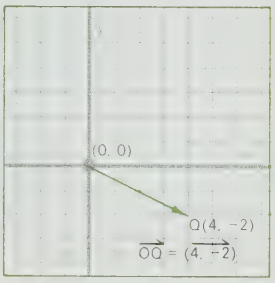
A vector is called a *position vector* if the initial point is (0, 0). Therefore, if $\overrightarrow{OP}$ is described by $\overrightarrow{(x, y)}$, its initial point will be (0, 0) and its end point (x, y).

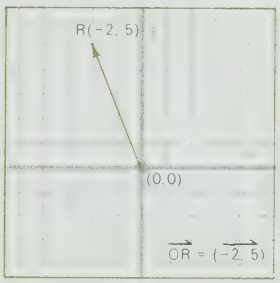


By choosing suitable values of x and y we can represent any vector in this way.

Mathematicians have agreed to name vectors in the following way.







Often the arrow is left off and $\overrightarrow{(4, 3)}$ is written vector (4, 3). For our work we shall always retain the arrow.

Example

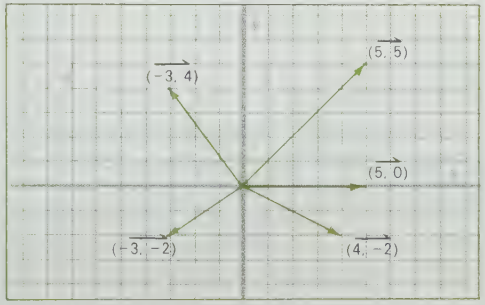
Make a diagram for the following position vectors.

(*a*) $\overrightarrow{(5, 5)}$ (*d*) $\overrightarrow{(-3, -2)}$

(*b*) $\overrightarrow{(0, 3)}$ (*e*) $\overrightarrow{(4, -2)}$

(*c*) $\overrightarrow{(-3, 4)}$ (*f*) $\overrightarrow{(5, 0)}$

Solution



$\overrightarrow{(-3, 4)}$ represents a vector; whereas (−3, 4) represents a point.

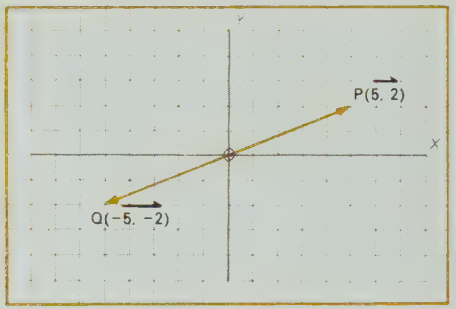
Now try these.

1. Draw the position vector equal to each given vector.
   (*a*) from A(1, 4) to B(4, 7)
   (*b*) from D(−2, 3) to E(−5, 8)
   (*c*) from F(−2, −4) to G(5, −6)

2. On the same diagram, draw and label the following position vectors.
   (*a*) $\overrightarrow{(4, 7)}$ (*b*) $\overrightarrow{(-4, -5)}$ (*c*) $\overrightarrow{(5, -6)}$

B Opposite vectors can also be described by ordered pairs.

$\overrightarrow{OP} = \overrightarrow{(5, 2)}$
and $\overrightarrow{OQ} = \overrightarrow{(-5, -2)}$
Also $\overrightarrow{OP} = -\overrightarrow{(OQ)}$
$= -\overrightarrow{(-5, -2)}$
$= \overrightarrow{(5, 2)}$
and $\overrightarrow{OQ} = -\overrightarrow{(OP)}$
$= -\overrightarrow{(5, 2)}$
$= \overrightarrow{(-5, -2)}$



$\overrightarrow{OP}$ is the opposite vector of $\overrightarrow{OQ}$.
$\overrightarrow{(5, 2)}$ is the opposite vector of $\overrightarrow{(-5, -2)}$.

1. State opposite vectors of each of the following.
   (*a*) $\overrightarrow{(2, 3)}$ (*c*) $\overrightarrow{(-2, 4)}$
   (*b*) $\overrightarrow{(-4, -5)}$ (*d*) $\overrightarrow{(5, -1)}$

2. Make a graph for each of the vectors in Question 1 and label the vector and its opposite vector.

Exercise 24.2

Written Solutions

1. On the same diagram, graph $\overrightarrow{(1, 4)}$ with each of the following given points as the initial point.
   (*a*) A(4, 1) (*c*) C(−3, 2) (*e*) E(−1, −6)
   (*b*) B(0, 0) (*d*) D(−1, 5) (*f*) F(4, −2)

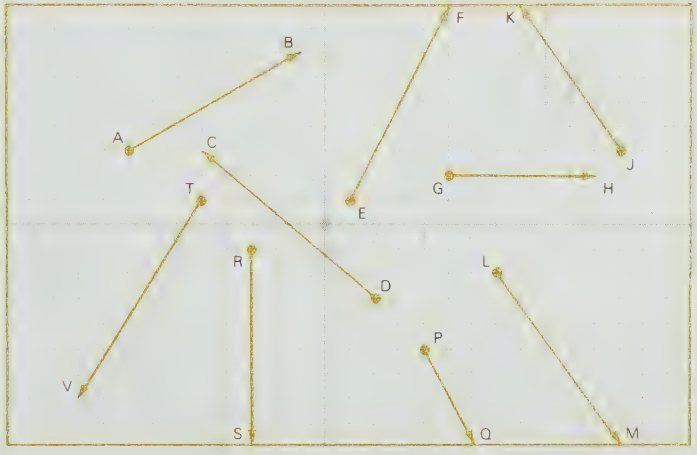
2. Express $\overrightarrow{AB}$ as an ordered pair.

(*a*)	A(0, 0), B(7, 5)	(*g*)	A(−1, −4), B(2, 1)
(*b*)	A(1, 2), B(2, 3)	(*h*)	A(1, −5), B(4, 2)
(*c*)	A(−1, −2), B(−1, 6)	(*i*)	A(7, 2), B(1, 6)
(*d*)	A(−3, −4), B(−2, 4)	(*j*)	A(−1, −1), B(−6, 2)
(*e*)	A(3, −2), B(6, 4)	(*k*)	A(1, −3), B(−1, 7)
(*f*)	A(−5, 0), B(5, 0)	(*l*)	A(1, 0), B(−4, 0)

3. Using the results of Question 2 above, complete the following table.

	A	B	*Run* A *to* B	*Rise* A *to* B	(*Run*, *Rise*)	$\overrightarrow{AB}$
(*a*)	(0, 0)	(7, 5)	7 − 0 = 7	5 − 0 = 5	(7, 5)	$\overrightarrow{(7, 5)}$
(*b*)	(1, 2)	(2, 3)	2 − 1 = 1	3 − 2 = 1	(1, 1)	$\overrightarrow{(1, 1)}$
.						
.						
.						
(*l*)						

4. Express each of the vectors shown in the following diagram as an ordered pair.



5. (*a*) Plot the points A(2, 4), B(7, 2), C(−3, −6), D(5, −1) and find the vectors represented by $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$, $\overrightarrow{AC}$, $\overrightarrow{DB}$.

(*b*) State the equivalent position vectors of each vector in part (*a*).

(*c*) State the opposite position vectors of each vector in part (*b*).

2.(a) $\overrightarrow{(7, 5)}$ (f) $\overrightarrow{(10, 0)}$
(b) $\overrightarrow{(1, 1)}$ (i) $\overrightarrow{(-6, 4)}$
(e) $\overrightarrow{(3, 6)}$ (j) $\overrightarrow{(-5, 3)}$

4. $\overrightarrow{AB} = \overrightarrow{(7, 4)}$ $\overrightarrow{RS} = \overrightarrow{(0, -8)}$
$\overrightarrow{CD} = \overrightarrow{(7, -6)}$ $\overrightarrow{TV} = \overrightarrow{(-5, -8)}$

5.(a) $\overrightarrow{AB} = \overrightarrow{(5, -2)}$ $\overrightarrow{DA} = \overrightarrow{(-3, 5)}$
$\overrightarrow{BA} = \overrightarrow{(-5, 2)}$ $\overrightarrow{AC} = \overrightarrow{(-5, -10)}$

6. Draw the position vector of each of the following.
   (*a*) $\overrightarrow{AB} = \overrightarrow{(-4, 3)}$
   (*b*) $\overrightarrow{CD} = \overrightarrow{(5, 0)}$
   (*c*) $\overrightarrow{EF} = \overrightarrow{(0, 4)}$
   (*d*) $\overrightarrow{GH} = \overrightarrow{(5, 7)}$
   (*e*) $\overrightarrow{IJ} = \overrightarrow{(-4, -5)}$

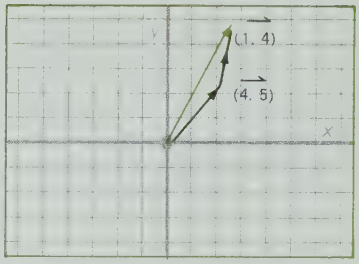
7. Express the following as vectors in ordered pairs.
   (*a*) 50 km/h north
   (*b*) 100 km east
   (*c*) 200 km northeast
   (*d*) 70 km/h south

7.(a) $\overrightarrow{(0, 50)}$ (b) $\overrightarrow{(100, 0)}$

24.3 Addition of Vectors

Just as we can add numbers of different magnitudes and obtain a result with a definite magnitude, we can also add vectors and obtain a resultant that has magnitude and direction. Suppose a plane makes a flight described by $\overrightarrow{(4, 5)}$ followed by $\overrightarrow{(1, 4)}$. This can be shown on a diagram.

The result of this journey could be described by the vector joining the initial point to the final point.



$\therefore$ the resultant journey can be described by $\overrightarrow{(4, 5)} + \overrightarrow{(1, 4)}$ or $\overrightarrow{(5, 9)}$.

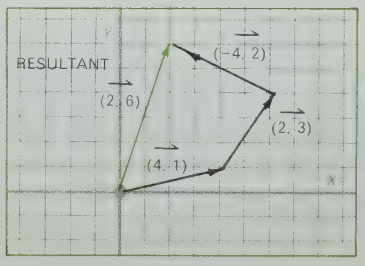
We can obtain a resultant vector by adding the x and y components of each individual vector.

$$\overrightarrow{(4, 5)} + \overrightarrow{(1, 4)} = \overrightarrow{(4 + 1, 5 + 4)}$$
$$= \overrightarrow{(5, 9)}$$
$\therefore$ the resultant is $\overrightarrow{(5, 9)}$.

Example

Find graphically the resultant vector of $\overrightarrow{(4, 1)}$, $\overrightarrow{(2, 3)}$, $\overrightarrow{(-4, 2)}$.

Solution



To describe the resultant using an ordered pair, add the x and y components of each vector.

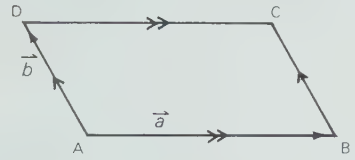
$$\therefore \overrightarrow{(4, 1)} + \overrightarrow{(2, 3)} + \overrightarrow{(-4, 2)} = \overrightarrow{(4 + 2 - 4, 1 + 3 + 2)}$$
$$= \overrightarrow{(2, 6)}$$
$$\therefore \text{the resultant is } \overrightarrow{(2, 6)}.$$

Exercise 24.3

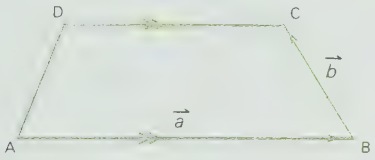
Written Solutions

1. In each diagram, find a vector that is equal to $\vec{a} + \vec{b}$.

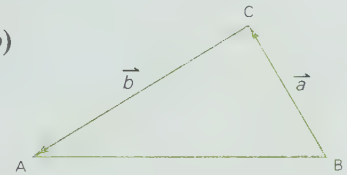
(a)



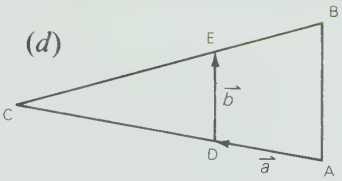
(c)



(b)



(d)



2. Add the following vectors and check by graphing.

(a) $\overrightarrow{(3, 4)}, \overrightarrow{(1, 2)}$
(b) $\overrightarrow{(-2, 4)}, \overrightarrow{(-4, -2)}$
(c) $\overrightarrow{(4, -3)}, \overrightarrow{(-3, 1)}$
(d) $\overrightarrow{(5, 6)}, \overrightarrow{(-1, -2)}$
(e) $\overrightarrow{(-3, -2)}, \overrightarrow{(5, 2)}$
(f) $\overrightarrow{(0, 0)}, \overrightarrow{(-4, 5)}$
(g) $\overrightarrow{(5, 2)}, \overrightarrow{(0, 0)}$
(h) $\overrightarrow{(-4, -7)}, \overrightarrow{(-5, -2)}$
(i) $\overrightarrow{(4, -2)}, \overrightarrow{(3, 5)}$
(j) $\overrightarrow{(2, -1)}, \overrightarrow{(-6, 5)}$

3. Find the sum of the following vectors.

(a) $\overrightarrow{(6, 5)}, \overrightarrow{(2, 7)}$
(b) $\overrightarrow{(1, 1)}, \overrightarrow{(0, 1)}$
(c) $\overrightarrow{(7, 2)}, \overrightarrow{(-2, -5)}$
(d) $\overrightarrow{(-2, 3)}, \overrightarrow{(6, -2)}$
(e) $\overrightarrow{(0, 3)}, \overrightarrow{(4, 0)}, \overrightarrow{(3, 6)}$
(f) $\overrightarrow{(2, 1)}, \overrightarrow{(4, 1)}, \overrightarrow{(-5, -6)}$
(g) $\overrightarrow{(2, 1)}, \overrightarrow{(0, 5)}, \overrightarrow{(-4, 1)}$
(h) $\overrightarrow{(a, b)}, \overrightarrow{(c, d)}$

4. Find the vector $\overrightarrow{(x, y)}$ for each of the following.

(a) $\overrightarrow{(x, y)} + \overrightarrow{(3, -7)} = \overrightarrow{(0, 0)}$
(b) $\overrightarrow{(-9, 4)} + \overrightarrow{(x, y)} = \overrightarrow{(-9, 4)}$
(c) $\overrightarrow{(4, -3)} + \overrightarrow{(x, y)} = \overrightarrow{(8, -6)}$
(d) $\overrightarrow{(3, 1)} + \overrightarrow{(-5, -6)} = \overrightarrow{(x, y)}$
(e) $\overrightarrow{(x, y)} + \overrightarrow{(4, 7)} = \overrightarrow{(9, 2)}$
(f) $\overrightarrow{(x, y)} + \overrightarrow{(2, 1)} = \overrightarrow{(-5, -7)}$

It's no more difficult than adding positive and negative numbers.

1.(a) $\overrightarrow{AC}$ (d) $\overrightarrow{AE}$
2.(a) $\overrightarrow{(4,6)}$ (i) $\overrightarrow{(7,3)}$
(b) $\overrightarrow{(-6,2)}$ (j) $\overrightarrow{(-4,4)}$
3.(b) $\overrightarrow{(1,2)}$ (f) $\overrightarrow{(1,-4)}$
(c) $\overrightarrow{(5,-3)}$ (g) $\overrightarrow{(-2,7)}$
4.(a) $\overrightarrow{(-3,7)}$ (d) $\overrightarrow{(-2,-5)}$

5. Using a suitable scale, find the resultant of the following vectors using a diagram.
   (*a*) 5 km/h east followed by 12 km/h south
   (*b*) 4 km/h north followed by 3 km/h west
   (*c*) 30 km/h south followed by 15 km/h north
   (*d*) 5 km/h west, 4 km/h north, 10 km/h east

24.4 Multiplication of a Vector by a Scalar

Now that we have found that adding vectors really involves only the addition of real numbers, let's find out what happens when we multiply a vector by a real number.

Discovery Project

1. Two cars leave the same starting point. The Ford travels according to $\overrightarrow{AB} = \overrightarrow{(5, 12)}$. The Pontiac travels according to $\overrightarrow{AC} = \overrightarrow{(10, 24)}$.
   (*a*) Make a diagram to show these vectors as ordered pairs.
   (*b*) Find $\overrightarrow{BC}$ so that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.
   (*c*) How are $\overrightarrow{AB}$ and $\overrightarrow{BC}$ related?
   (*d*) How are $\overrightarrow{AB}$ and $\overrightarrow{AC}$ related?
   (*e*) State the relationship between $\overrightarrow{BC}$ and $\overrightarrow{AC}$.
2. A sailboat sails according to the vector $\vec{a} = \overrightarrow{(3, 4)}$.
   (*a*) Find $\vec{b}$, so that $\vec{b} = 2\vec{a}$.
   (*b*) How are the lengths of $\vec{a}$ and $\vec{b}$ related?
   (*c*) How are the directions of $\vec{a}$ and $\vec{b}$ related?

$k(\overrightarrow{x,y}) = (\overrightarrow{kx,ky})$

3. State the effect on the magnitude and direction of $\overrightarrow{(x, y)}$ by multiplying by each of the following.
   (*a*) 3 (*b*) -3 (*c*) $+a$ (*d*) $-a$
4. Given $\vec{m} = \overrightarrow{(4, 1)}$.
   (*a*) Find $2\vec{m}$, $-\vec{m}$, $-3\vec{m}$, $3\vec{m}$.
   (*b*) Draw $\vec{m}$, $2\vec{m}$, $-\vec{m}$, $-2\vec{m}$.
5. Given $\vec{a} = \overrightarrow{(-3, 1)}$ and the scalar quantity x where $x = 2$, express each vector as an ordered pair.
   (*a*) $x\overrightarrow{(-3, 1)}$ (*d*) $-\frac{1}{2}x\overrightarrow{(-3, 1)}$
   (*b*) $2x\overrightarrow{(-3, 1)}$ (*e*) $-3x\overrightarrow{(-3, 1)}$
   (*c*) $\frac{x}{3}\overrightarrow{(-3, 1)}$ (*f*) $-\frac{x}{4}\overrightarrow{(-3, 1)}$

Exercise 24.4

1. State each of the following vectors in the form $\overrightarrow{(x, y)}$.
   (a) $3\overrightarrow{(1, 2)}$
   (b) $\frac{1}{2}\overrightarrow{(6, 8)}$
   (c) $-2\overrightarrow{(-3, -2)}$
   (d) $-7\overrightarrow{(2, -1)}$
   (e) $\frac{1}{4}\overrightarrow{(8, 2)}$
   (f) $\frac{2}{3}\overrightarrow{(-6, -9)}$
   (g) $-\frac{1}{3}\overrightarrow{(-7, 3)}$
   (h) $-\frac{1}{5}\overrightarrow{(10, 2)}$
   (i) $4\overrightarrow{(3, -2)}$
   (j) $\frac{7}{2}\overrightarrow{(4, 2)}$

2. Simplify each of the following and express the answer in the form $\overrightarrow{(x, y)}$.
   (a) $3\overrightarrow{(1, 2)} + 2\overrightarrow{(3, 1)}$
   (b) $\frac{1}{2}\overrightarrow{(0, 2)} + 4\overrightarrow{(1, 0)}$
   (c) $-2\overrightarrow{(2, 7)} + (-3)\overrightarrow{(7, -2)} + 4\overrightarrow{(1, 2)}$
   (d) $3\overrightarrow{(1, 2)} + \overrightarrow{(2, 3)}$
   (e) $-\frac{1}{3}\overrightarrow{(3, -6)} + (-\frac{1}{2})\overrightarrow{(4, 2)}$
   (f) $2\overrightarrow{(-1, 5)} + \overrightarrow{(3, -2)}$
   (g) $\frac{1}{2}\overrightarrow{(4x, 4y)} + (-2)\overrightarrow{(x, y)}$

3. If a is 2 units in magnitude and directed to the right, illustrate each of the following vectors.
   (a) $3\vec{a}$
   (b) $-\vec{a}$
   (c) $\frac{1}{2}\vec{a}$
   (d) $2\vec{a}$
   (e) $-2\vec{a}$
   (f) $\frac{1}{6}\vec{a}$
   (g) $4\vec{a}$
   (h) $\vec{a} + \frac{1}{2}\vec{a} + \frac{3}{2}\vec{a}$
   (i) $-2\vec{a} + \frac{1}{2}\vec{a}$
   (j) $-\frac{3}{4}\vec{a} + \frac{5}{2}\vec{a}$

4. Calculate the value of $\overrightarrow{(x, y)}$ in each of the following.
   (a) $3\overrightarrow{(2, 1)} + \overrightarrow{(x, y)} = \overrightarrow{(4, 6)}$
   (b) $2\overrightarrow{(x, y)} + \overrightarrow{(4, -1)} = \overrightarrow{(8, -3)}$
   (c) $\frac{1}{2}\overrightarrow{(4, -2)} + \overrightarrow{(x, y)} = \overrightarrow{(7, 1)}$
   (d) $\frac{1}{3}\overrightarrow{(-6, -3)} + \frac{3}{4}\overrightarrow{(4, 4)} = \overrightarrow{(x, y)}$
   (e) $-1\overrightarrow{(4, -2)} + \frac{1}{2}\overrightarrow{(x, y)} = \overrightarrow{(3, 5)}$

5. If $\vec{x} = \overrightarrow{(a, b)}$ and $\vec{y} = \overrightarrow{(2a, 4b)}$, find the following.
   (a) $2\vec{x} + \frac{1}{2}\vec{y}$
   (b) $3\vec{x} + 2\vec{y}$
   (c) $\frac{1}{2}\vec{x} + \frac{3}{2}\vec{y}$
   (d) $-\vec{x} + (-\frac{1}{2})\vec{y}$
   (e) $\vec{x} - \vec{y}$

$(-\frac{1}{3})\overrightarrow{(3,-6)} + (-\frac{1}{2})\overrightarrow{(4,2)}$
$= \overrightarrow{(-1, 2)} + \overrightarrow{(-2,-1)}$
$= \overrightarrow{(-1-2, 2-1)}$
$= \overrightarrow{(-3, 1)}$

1.(a) $\overrightarrow{(3, 6)}$ (i) $\overrightarrow{(12, -8)}$
(b) $\overrightarrow{(3, 4)}$ (j) $\overrightarrow{(14, 7)}$
2.(a) $\overrightarrow{(9, 8)}$ (e) $\overrightarrow{(-3, 1)}$
(b) $\overrightarrow{(4, 1)}$ (f) $\overrightarrow{(1, 8)}$
4.(a) $\overrightarrow{(-2, 3)}$ (d) $\overrightarrow{(1, 2)}$
5.(a) $\overrightarrow{(3a, 4b)}$ (d) $\overrightarrow{(-2a, -3b)}$

24.5 Subtraction of Vectors

The opposite of vector $(\overrightarrow{3,2})$ is $(\overrightarrow{-3,-2})$ since $-(\overrightarrow{3,2}) = (\overrightarrow{-3,-2})$

Recall that to subtract a real number, you "add the opposite."

$$7 - 4 = 7 + (-4) = 3$$
$$8 - (-4) = 8 + (+4) = 12$$
$$-9 - 5 = -9 + (-5) = -14$$

Subtraction of vectors is defined in the same way as subtraction of real numbers, that is, "to subtract a vector, add the opposite vector."

Example 1

Simplify $\overrightarrow{(7, 5)} - \overrightarrow{(3, -5)}$.

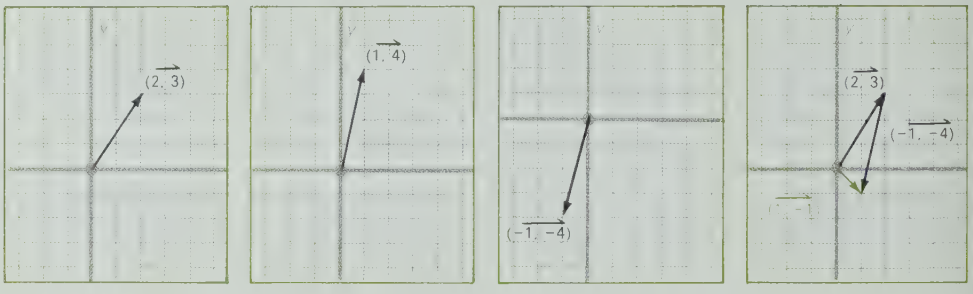
Solution

$$\overrightarrow{(7, 5)} - \overrightarrow{(3, -5)} = \overrightarrow{(7, 5)} + \overrightarrow{(-3, 5)}$$
$$= \overrightarrow{(4, 10)}$$

Example 2

Show on a diagram that $\overrightarrow{(2, 3)} - \overrightarrow{(1, 4)} = \overrightarrow{(1, -1)}$

Solution



$$\overrightarrow{(2, 3)} - \overrightarrow{(1, 4)} = \overrightarrow{(2, 3)} + \overrightarrow{(-1, -4)} = \overrightarrow{(1, -1)}$$

Exercise 24.5

1. Simplify and express your answer in the form $\overrightarrow{(x, y)}$.

(a) $\overrightarrow{(5, 12)} - \overrightarrow{(-5, -12)}$
(b) $\overrightarrow{(5, 2)} - \overrightarrow{(4, 3)}$
(c) $\overrightarrow{(-7, -2)} - \overrightarrow{(5, 6)}$
(d) $\overrightarrow{(-3, 4)} - \overrightarrow{(-5, -6)}$
(e) $\overrightarrow{(4, 5)} - \overrightarrow{(-2, 3)}$
(f) $\overrightarrow{(1, -2)} - \overrightarrow{(4, 1)}$
(g) $\overrightarrow{(-2, 8)} - \overrightarrow{(-5, 6)}$
(h) $\overrightarrow{(7, 1)} - \overrightarrow{(-3, -2)}$
(i) $\overrightarrow{(-1, 2)} - \overrightarrow{(4, 1)}$
(j) $\overrightarrow{(8, 11)} - \overrightarrow{(51, -21)}$

1.(a) $\overrightarrow{(10, 24)}$ (i) $\overrightarrow{(-5, 1)}$
(b) $\overrightarrow{(1, -1)}$ (j) $\overrightarrow{(-43, 32)}$

2. Simplify and express your answer in the form $\overrightarrow{(x, y)}$.

(*a*) $-5\overrightarrow{(6, -3)} - \overrightarrow{(-2, 5)}$ (*e*) $4\overrightarrow{(1, 1)} - \frac{1}{3}\overrightarrow{(3, 3)}$

(*b*) $-2\overrightarrow{(4, -1)} - \overrightarrow{(4, 8)}$ (*f*) $2\overrightarrow{(5, 2)} - 3\overrightarrow{(-1, -2)}$

(*c*) $3\overrightarrow{(5, 2)} - 7\overrightarrow{(1, -1)}$ (*g*) $-\frac{1}{2}\overrightarrow{(-4, -2)} - \frac{1}{4}\overrightarrow{(-8, -4)}$

(*d*) $2\overrightarrow{(3, 2)} - 3\overrightarrow{(1, 2)}$ (*h*) $-2\overrightarrow{(-1, -1)} - 2\overrightarrow{(8, 4)}$

3. Given $\overrightarrow{AB} = \overrightarrow{(-1, -4)}$, $\overrightarrow{CD} = \overrightarrow{(2, -3)}$ and $\overrightarrow{EF} = \overrightarrow{(3, 1)}$, find the following vectors and illustrate each with a diagram.

(*a*) $\overrightarrow{AB} - \overrightarrow{CD}$ (*d*) $\overrightarrow{EF} - \overrightarrow{CD}$

(*b*) $\overrightarrow{AB} - \overrightarrow{EF}$ (*e*) $\overrightarrow{EF} - \overrightarrow{AB}$

(*c*) $\overrightarrow{CD} - \overrightarrow{EF}$ (*f*) $\overrightarrow{CD} - \overrightarrow{AB}$

4. Make a diagram and express each of the following as a single vector.

(*a*) $\overrightarrow{(3, 2)} - \overrightarrow{(-5, -3)}$ (*f*) $\overrightarrow{(-3, 12)} - \overrightarrow{(2, -7)}$

(*b*) $\overrightarrow{(-4, 3)} - \overrightarrow{(2, -1)}$ (*g*) $3\overrightarrow{(2, -5)} - 3\overrightarrow{(-5, 2)}$

(*c*) $\overrightarrow{(-5, -2)} - \overrightarrow{(2, -3)}$ (*h*) $2\overrightarrow{(4, 1)} - 3\overrightarrow{(1, -2)}$

(*d*) $\overrightarrow{(1, -1)} - \overrightarrow{(4, 6)}$ (*i*) $\overrightarrow{(3, -7)} - 4\overrightarrow{(0, 2)}$

(*e*) $\overrightarrow{(-3, 4)} - \overrightarrow{(2, 1)}$ (*j*) $\overrightarrow{(7, -2)} - \overrightarrow{(2, 5)} - \overrightarrow{(-1, -1)}$

5. Make a diagram to show that each of the following is true.

(*a*) $3[\overrightarrow{(1, 2)} - \overrightarrow{(2, 1)}] = 3\overrightarrow{(1, 2)} - 3\overrightarrow{(2, 1)}$

(*b*) $\frac{1}{2}\overrightarrow{(4, 2)} - \frac{1}{2}\overrightarrow{(-8, -4)} = \overrightarrow{(2, 1)} + \overrightarrow{(4, 2)}$

(*c*) $3[\overrightarrow{(2, 4)} - \overrightarrow{(3, 7)}] = 3\overrightarrow{(2, 4)} - 3\overrightarrow{(3, 7)}$

6. Solve for $\overrightarrow{(x, y)}$ in each of the following.

(*a*) $\overrightarrow{(3, 4)} - \overrightarrow{(x, y)} = \overrightarrow{(2, 4)}$

(*b*) $\overrightarrow{(-3, -8)} - \overrightarrow{(-2, -4)} = \overrightarrow{(x, y)}$

(*c*) $2\overrightarrow{(1, 1)} - \frac{1}{2}\overrightarrow{(2, -4)} = \overrightarrow{(x, y)}$

(*d*) $\frac{1}{2}\overrightarrow{(8, 4)} - 2\overrightarrow{(-2, -1)} = \overrightarrow{(x, y)}$

(*e*) $\frac{1}{2}\overrightarrow{(x, y)} - 2\overrightarrow{(-3, -4)} = \overrightarrow{(2, 1)}$

7. State the resultant of each of the following in the form $\overrightarrow{(x, y)}$.

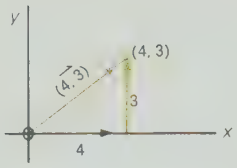
(*a*) $\overrightarrow{(0, 2)} - \overrightarrow{(3, 3)}$

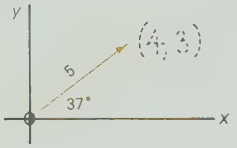
(*b*) $\overrightarrow{(1, 4)} - \overrightarrow{(-2, -3)}$

(*c*) $\overrightarrow{(1, -2)} - 2\overrightarrow{(1, -1)}$

(*d*) $\overrightarrow{(5, 0)} - 3\overrightarrow{(1, -2)}$

2.(a) $\overrightarrow{(-28, 10)}$ (e) $\overrightarrow{(3, 3)}$
(d) $\overrightarrow{(3, -2)}$ (h) $\overrightarrow{(-14, -6)}$
3.(b) $\overrightarrow{(-4, -5)}$ (f) $\overrightarrow{(3, 1)}$
(c) $\overrightarrow{(-1, -4)}$
4.(b) $\overrightarrow{(-6, 4)}$ (g) $\overrightarrow{(21, -21)}$
(c) $\overrightarrow{(-7, 1)}$ (j) $\overrightarrow{(6, -6)}$
(f) $\overrightarrow{(-5, 19)}$
6.(a) $\overrightarrow{(1, 0)}$ (b) $\overrightarrow{(-1, -4)}$
7.(a) $\overrightarrow{(-3, -1)}$ (b) $\overrightarrow{(3, 7)}$

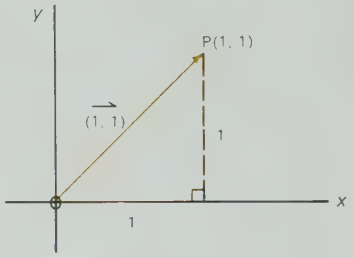




24.6 Vectors and Trigonometry

A We know that vectors have magnitude and direction and can be described by means of an ordered pair. Another method of describing a vector is in terms of its magnitude and angle of rotation from some reference axis.

$\overrightarrow{OP}$ can be described by $(\overrightarrow{1, 1})$. In order to find its magnitude we must use the Pythagorean theorem.



$$OP^2 = OT^2 + PT^2$$
$$\therefore OP^2 = 1^2 + 1^2$$
$$= 2$$
$$\therefore OP^2 = \sqrt{2}$$

$\therefore$ the magnitude of $\overrightarrow{OP} = \sqrt{2}$.

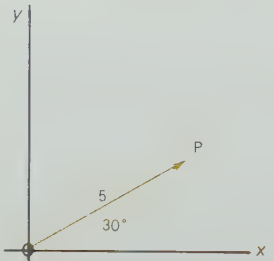
To find the angle of rotation of the vector from the horizontal axis, we must find the tan of $\angle$ POT.

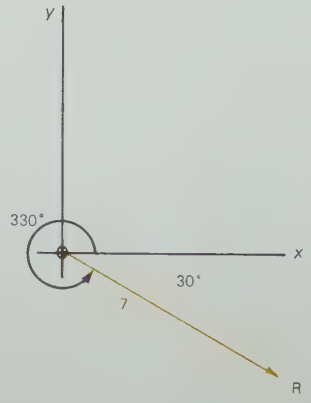
$$\because \tan \angle POT = \frac{1}{1}$$
$$\therefore \angle POT = 45°$$

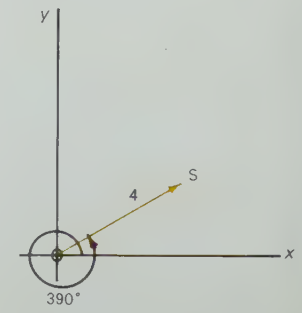
Therefore a vector can be described by the notation $\overrightarrow{OP} = \sqrt{2}/\underline{45°}$, where $\sqrt{2}$ indicates the magnitude and 45° the angle of rotation from the horizontal axis.

In general terms, if r represents the magnitude and θ the angle of rotation, $r/\underline{\theta}$ describes any vector.

1. Describe each of the following vectors in the form $r/\underline{\theta}$.







$4/\underline{390°}$
$= 4/\underline{30°}$

B Now let's try the problem in the opposite way. If $2\angle 30°$ describes $\overrightarrow{OP}$, state the ordered pair that identifies OP as a position vector.

Since $\frac{x}{2} = \cos 30°$

and $\frac{y}{2} = \sin 30°$

$\therefore x = 2 \cos 30°$

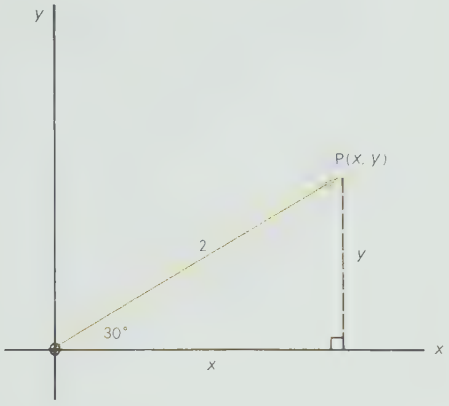
$= 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

and $y = 2 \sin 30°$

$= 2\left(\frac{1}{2}\right)$

$= 1$

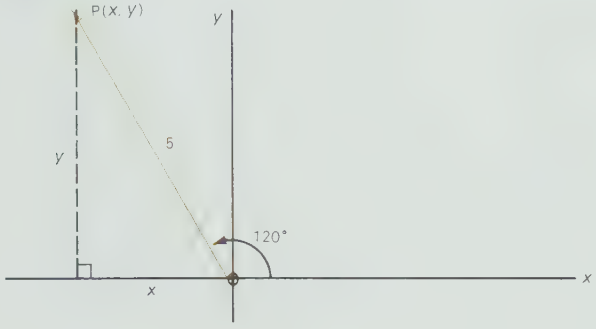
$\therefore \overrightarrow{OP} = (\overrightarrow{\sqrt{3}, 1})$



Example 1

Find the ordered pair that describes vector $5\angle 120°$.

Solution



Since $\cos 120° = \cos (180° - 60°)$

$= -\cos 60°$

$\therefore \frac{x}{5} = -\cos 60°$ and $\frac{y}{5} = \sin (180° - 60°)$

$\therefore x = 5\left(-\frac{1}{2}\right)$ $= \sin 60°$

$= -\frac{5}{2}$ $\therefore y = 5\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2}$

$\therefore \overrightarrow{OP} = \left(\overrightarrow{-\frac{5}{2}, \frac{5\sqrt{3}}{2}}\right)$

Example 2

If $\overrightarrow{OP} = 2\angle -30°$, describe $\overrightarrow{OP}$ in terms of an ordered pair.

Solution

Since $\cos(-30°) = \cos 30°$
and $\sin(-30°) = -\sin 30°$

$$\therefore \frac{x}{2} = \cos 30°$$

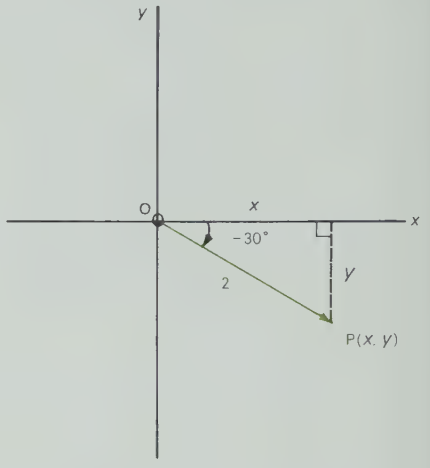
$$\therefore x = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$\text{Also } \frac{y}{2} = -\sin 30°$$

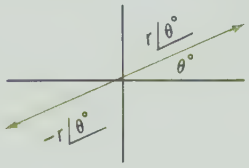
$$\therefore y = 2\left(-\frac{1}{2}\right)$$

$$= -1$$

$$\therefore \overrightarrow{OP} = (\sqrt{3}, -1)$$



Notice that $r\angle\theta°$ and $-r\angle\theta°$ are opposite.



Example 3

If $-4\angle 120°$ describes $\overrightarrow{OP}$, find the ordered pair that also describes $\overrightarrow{OP}$.

Solution

$$\therefore \frac{x}{-4} = \cos 120°$$

$$\therefore x = -4(-\cos 60°)$$

$$= -4\left(-\frac{1}{2}\right) = 2$$

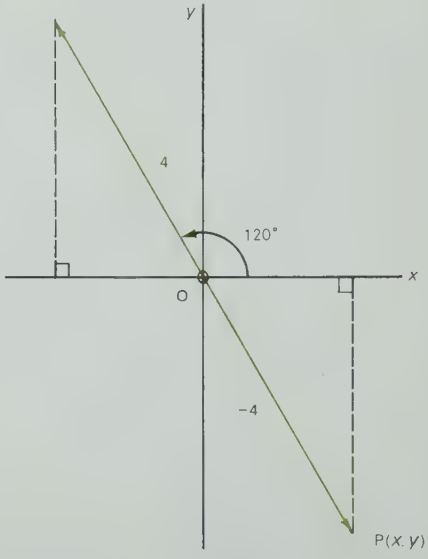
$$\text{Also } \frac{y}{-4} = \sin 120°$$

$$\therefore y = -4(\sin 60°)$$

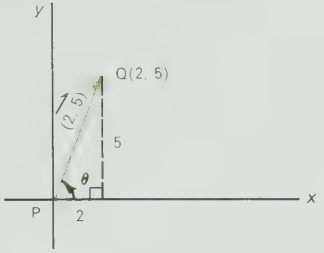
$$= -4\left(\frac{\sqrt{3}}{2}\right)$$

$$= -2\sqrt{3}$$

$$\therefore OP = (2, -2\sqrt{3})$$



C If $\overrightarrow{PQ} = \overrightarrow{(2, 5)}$, describe $\overrightarrow{PQ}$ in terms of its magnitude and angle of rotation.



Since we know $\tan\theta = \dfrac{5}{2} = 2.5$

$\therefore \theta = 68°$ (from tables)

Also $r^2 = 2^2 + 5^2$

$= 29$

$\therefore r = \sqrt{29}$

$\therefore \overrightarrow{PQ} = \sqrt{29}/\underline{68°}$

Exercise 24.6

1. Express each of the following vectors in the form $\overrightarrow{(x, y)}$.
   (*a*) $12/\underline{25°}$ (*f*) $3/\underline{-270°}$
   (*b*) $10/\underline{30°}$ (*g*) $-7/\underline{-90°}$
   (*c*) $4/\underline{45°}$ (*h*) $-5/\underline{180°}$
   (*d*) $4/\underline{-60°}$ (*i*) $2/\underline{135°}$
   (*e*) $2/\underline{225°}$ (*j*) $-3/\underline{330°}$

2. Describe the following vectors in the form $r/\underline{\theta°}$.
   (*a*) $\overrightarrow{(3, 6)}$ (*f*) $\overrightarrow{(0, 2)}$
   (*b*) $\overrightarrow{(-4, 4)}$ (*g*) $\overrightarrow{(4, -3)}$
   (*c*) $\overrightarrow{(3, 1)}$ (*h*) $\overrightarrow{(0, -3)}$
   (*d*) $\overrightarrow{\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)}$ (*i*) $\overrightarrow{(-3, 4)}$
   (*e*) $\overrightarrow{(5, 0)}$ (*j*) $\overrightarrow{(-6, 0)}$

3. Express each of the following vectors in the form $r/\underline{\theta.°}$
   (*a*) a velocity of 50 km/h south
   (*b*) a force of 20 N acting to the left
   (*c*) a 40 N force acting 30° to the horizontal
   (*d*) a force of 10 N acting down
   (*e*) a velocity of 30 km/h in the northeast direction

If $(\overrightarrow{x, y}) = r/\underline{\theta°}$

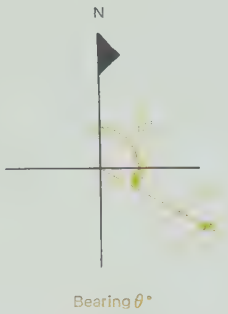
$x = r\cos\theta$

$y = r\sin\theta$

$\frac{y}{x} = \tan\theta$

$r = \sqrt{x^2 + y^2}$

1.(b) $(\overrightarrow{5\sqrt{3}, 5})$ (g) $(\overrightarrow{0, 7})$
(c) $(\overrightarrow{2\sqrt{2}, 2\sqrt{2}})$ (h) $(\overrightarrow{5, 0})$
2.(c) $\sqrt{10}\ /\underline{18°}$ (h) $3\ /\underline{270°}$
(d) $1\ /\underline{315°}$ (i) $5\ /\underline{127°}$
3.(a) $50\ /\underline{270°}$ (b) $20\ /\underline{180°}$
4. $10\sqrt{20}\ /\underline{63°}$ or $44.72\ /\underline{63°}$
6. $2\sqrt{3}\ /\underline{120°}$ or $3.464\ /\underline{120°}$



4. The velocity of a sailboat can be represented by the vector $\overrightarrow{(20, 40)}$. Calculate the magnitude and direction angle and describe as $r/\underline{\theta°}$.
5. A jetliner has an air speed of 500 km/h on a bearing of 45°. Express this vector as an ordered pair.
6. If $\overrightarrow{OP} = 2/\underline{30°}$ and $\overrightarrow{PQ} = 4/\underline{150°}$, find the sum of $\overrightarrow{OP} + \overrightarrow{PQ}$ and express the resultant in the form $r/\underline{\theta°}$.

24.7 Solving Problems with Vectors

Many practical problems can be solved by using vectors. Vectors can be used to explain problems of forces, velocities, electricity, and navigation.

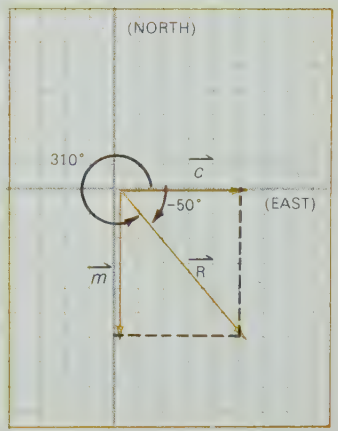
The main concept to remember is that the resultant of a system involving vectors is a single vector that will replace all the others and have the same effect.

Example 1

A motorboat is steered due south at a speed of 6 km/h. There is a water current, running from west to east, of 5 km/h. Find the actual speed and direction of the boat. Make an accurate scale diagram to illustrate your resultant.

Solution

Let $\vec{m}$ represent the motion of the motorboat and $\vec{c}$ represent the motion of the current.
$\therefore$ resultant vector, $\vec{R}$, will be $\vec{m} + \vec{c}$.



$$\text{Now } \vec{m} = \overrightarrow{(0, -6)}$$
$$\text{and } \vec{c} = \overrightarrow{(5, 0)}$$
$$\therefore \vec{R} = \vec{m} + \vec{c} = \overrightarrow{(5, -6)}$$
$$\therefore \text{actual speed} = |\vec{R}| = \sqrt{25 + 36}$$
$$= \sqrt{61}$$
$$\doteq 7.8 \text{ km/h}$$

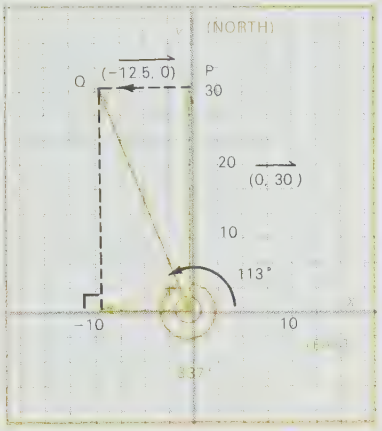
And the actual direction can be found by finding $\tan \theta$.

$$\tan \theta = \frac{-6}{5} = -1.2$$
$$\therefore \theta = -50° \text{ or } 310°$$
$$\therefore \vec{R} = 7.8/\underline{310°}$$

Example 2

Show on an accurate scale diagram, and also by calculation, the equivalent of a 30 km trip north and a 12.5 km trip west.

Solution



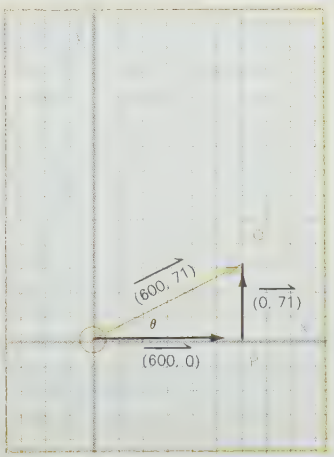
$$\overrightarrow{OP} = \overrightarrow{(0, 30)}$$
$$\overrightarrow{PQ} = \overrightarrow{(-12.5, 0)}$$
$$\therefore \overrightarrow{OQ} = \overrightarrow{(-12.5, 30)}$$
$$\tan\theta = \frac{30}{-12.5} = -2.4$$
$$\therefore \theta \doteq (180° - 67°)$$
$$= 113°$$
$$|\overrightarrow{OQ}| = \sqrt{(-12.5)^2 + (30)^2}$$
$$= \sqrt{156.25 + 900}$$
$$= \sqrt{1056.25}$$
$$\doteq 32.5$$
$$\therefore \overrightarrow{OQ} = 32.5\underline{/113°}$$

$\therefore$ the equivalent trip is 32.5 km, on a bearing of 337°.

Example 3

An airliner is headed due east at 600 km/h. A 71 km/h south wind is blowing. What is the resultant velocity and direction of the airliner?

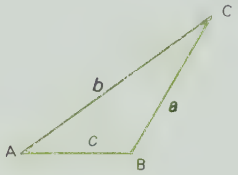
Solution



Since the resultant $= \overrightarrow{OQ}$
$$= \overrightarrow{OP} + \overrightarrow{PQ}$$
$$= \overrightarrow{(600, 0)} + \overrightarrow{(0, 71)}$$
$$= \overrightarrow{(600, 71)}$$
$$|\overrightarrow{OQ}| = \sqrt{(600)^2 + (71)^2}$$
$$= \sqrt{360\,000 + 5041}$$
$$= \sqrt{365\,041}$$
$$\doteq 604$$
$$\text{Also } \tan\theta = \frac{71}{600} \doteq 0.118$$
$$\therefore \theta = 7°$$
$$\therefore \overrightarrow{OQ} = 604\underline{/7°}$$

The resultant velocity of the plane is 604 km/h and the flight path is on a bearing of 83°.

Recall the Cosine Law



$b^2 = a^2 + c^2 - 2ac\cos B$

And the Sine Law

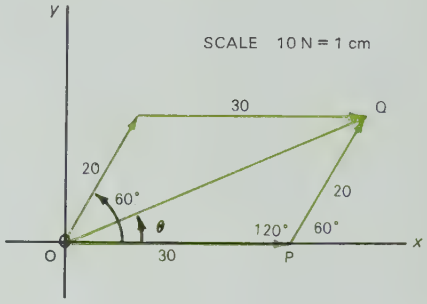
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Example 4

Two forces of 20 N and 30 N act on a single point. The angle between them is 60°. Find the magnitude and direction of the resultant force with the 30 N force.

Solution

$$\begin{aligned} |\vec{R}| &= \sqrt{OP^2 + PQ^2 - 2\,OP\,PQ\cos 120^\circ} \\ &= \sqrt{(30)^2 + (20)^2 - 2(30)(20)(-\tfrac{1}{2})} \\ &= \sqrt{900 + 400 + 600} \\ &= \sqrt{1900} \\ &\doteq 43.6 \end{aligned}$$



$$\text{Also } \frac{\sin\theta}{20} = \frac{\sin 120^\circ}{43.6}$$

$$\therefore \sin\theta = \frac{20}{43.6} \times \frac{\sqrt{3}}{2}$$

$$\doteq 0.397$$

$$\therefore \theta \doteq 23^\circ$$

$$\therefore \vec{R} = 43.6\angle 23^\circ$$

$\therefore$ the resultant force has a magnitude of 43.6 N acting at 23° from the 30 N force.

Exercise 24.7

1. Find the magnitude and direction of the resultant if a car travels 5 km east and then 12 km north. Check your answer by making an accurate scale diagram. State your answer in the form $(\vec{x}, \vec{y})$ and $r\angle\theta°$.
2. Two forces acting at 90° to each other act on an object with a force of 7 N each. Find the magnitude and direction of the resultant. Verify your answer with an accurate scale diagram and state your answer in the form $r\angle\theta°$.
3. A sailboat heads north across a lake at 10 km/h, propelled by a wind blowing from the west. The boat drifts toward the east at 5 km/h. Find the resultant speed and direction of the boat.
4. Pilot Pete Prangem sets his course at 400 km/h east. Find the plane's ground speed and direction of flight if a 45 km/h south wind is blowing.

1. $(5, \vec{12})$, 13 $\angle 67°$
3. 11.18 $\angle 63°$

5. An aircraft heads on a bearing of 60° at 200 km/h. The wind velocity is 50 km/h bearing 330°. Find the resultant velocity and direction of the aircraft.

6. A car travels 50 km on a bearing of 20° and then 40 km on a bearing of 60°. Find its resultant. Make an accurate scale diagram to verify your answer.

7. Find the resultant of two forces acting at an angle of 18° with each other. One force is 25 N and the other is 58 N. Verify your answer with a sketch.

8. Two forces of 70 N and 85 N have a resultant of 125 N. Find the angle which each of the forces makes with the resultant.

9. Find the horizontal and vertical forces that have as their resultant a force of 5680 N acting at an angle of 24° with the horizontal.

10. A force of 240 N is applied in pushing a go-cart with a stick 2 m long whose upper end is 1 m off the ground. Determine the force that acts to move the cart forward.

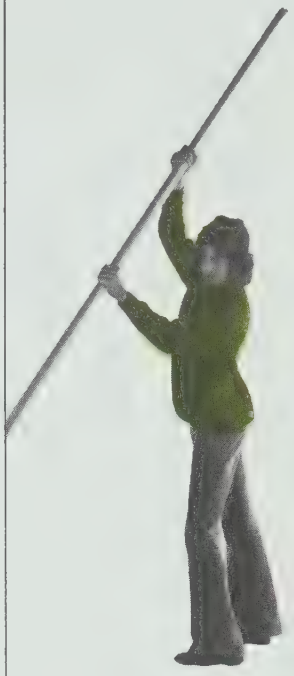
11. Phineas Phlatbottom can row a boat at 3 m/s in still water. If he rows upstream at an angle of 60° to the shore in a current of 2 m/s, what is his actual velocity? Sketch a diagram to verify your answer.

12. Pat Propwash is flying an aircraft with a speed of 250 km/h. She wants to fly due north and there is a west wind of 40 km/h. In what direction should she fly and what is her resultant speed?

13. Find the velocity an airplane with a speed of 200 km/h must have in order to travel due north if an east wind of 30 km/h is blowing. Verify your answer with a sketch.

14. Huck Finnberry is pushing a floating log with a long pole. The force exerted is 110 N and the pole makes an angle of 53° with the surface of the water. What force tends to submerge the log and what force tends to move it through the water?

15. Horace and Horatio are attempting to push a car out of a snowdrift by pushing as shown. Horace pushes with a force of 110 N, Horatio with a force of 260 N, and the angle between them is 18°. Find the resultant force on the car. Verify your answer with a sketch.



15.

5. 206 ∠44°
7. 82 N
9. 5189 N (horizontal), 2310 N (vertical)
11. 2.6 ∠101°
13. 198 ∠90°
15. 366 N

Unit 25
The Exponential Relation

25.1 Exponential Growth

It has been stated that "nearly all of mankind's current activities, from the use of fertilizer to the expansion of cities, can be represented by exponential growth curves."

from
The Limits to Growth
by Meadows, Meadows,
Randers, Behrens

In connection with this exponential growth pattern, U Thant, the Secretary-General of the United Nations in 1969 stated, "The Members of the United Nations have perhaps ten years left in which to subordinate their ancient quarrels and launch a global partnership to curb the arms race, to improve the human environment, to defuse the population explosion, and to supply the required momentum to development efforts. If such a global partnership is not forged within the next decade, then I very much fear that the problems I have mentioned will have reached such staggering proportions that they will be beyond our capacity to control."

These comments will indicate how very important the general nature of exponential growth is to us all.

A *Population Growth*

The present growth of world population is about 2.1 percent per year.

Thus, if the world population in 1972 is 3.30 billion people, the population in 1973 would be

$$\begin{aligned} &* 3.30 + (2.1\% \text{ of } 3.30) \\ &= 3.30 + 3.30 \times 0.021 \\ &= 3.30(1 + 0.021) \\ &= 3.30(1.021) \\ &\doteq 3.37 \text{ (billion)} \end{aligned}$$

and in 1974 it would be the population in 1973 + (2.1% of population in 1973)

$$\begin{aligned} &= 3.30(1.021) + 3.30(1.021) \times 0.021 \\ &= 3.30(1.021)(1 + 0.021) \\ &= 3.30(1.021)(1.021) \\ &= 3.30(1.021)^2 \\ &\doteq 3.44 \text{ (billion)} \end{aligned}$$

and in 1975, after *three years* it would be

$$\begin{aligned} &3.30(1 + 0.021)^3 \\ &\doteq 3.51 \text{ (billion)} \end{aligned}$$

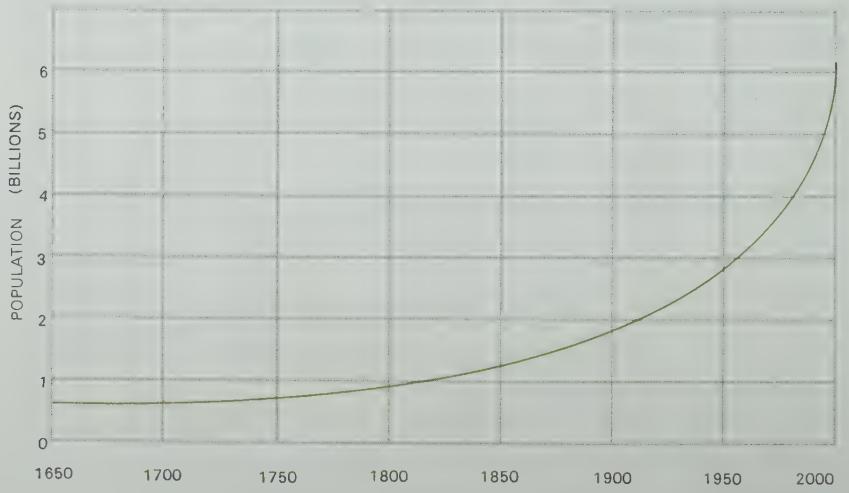
and in 1976, after *four years* it would be

$$\begin{aligned} &3.30(1 + 0.021)^4 \\ &\doteq 3.59 \text{ (billion)} \end{aligned}$$

and in 2000, after *twenty-eight years* it would be

$$\begin{aligned} &3.30(1 + 0.021)^{28} \\ &\doteq 5.91 \text{ (billion)} \end{aligned}$$

The graph shows this startling increase.



* $2.1\% = \frac{2.1}{100} = 0.021$

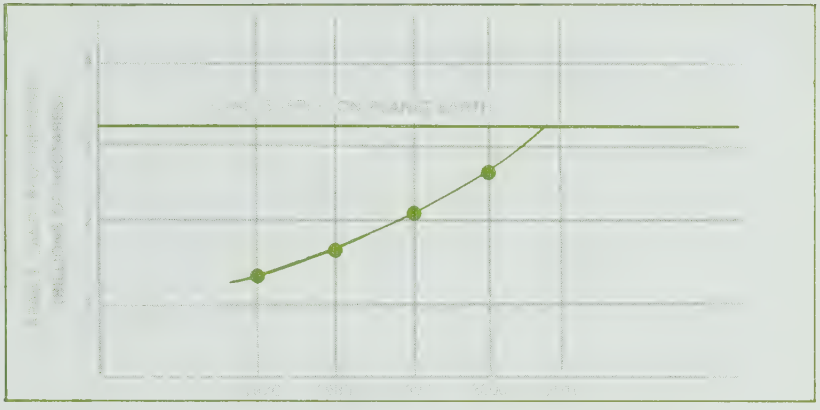
Recall that one billion = 10^9 (in North America).

After n years, $3.30(1+0.021)^n$

B *Arable Land*

At present day standards, the amount of arable land needed per person is approximately 0.45 ha (hectares).

Thus you can see that, as the population expands, the amount of arable land required to sustain life must also expand. The land requirements are shown in the graph. Note that a line is drawn across at 3.2 billion ha which is the total world supply of arable land.



What does the graph indicate for the year 2010?

What would you consider to be the situation if the amount of available land does not remain constant, but in fact declines due to the expansion of cities, roads, and industry?

C *Growth in Savings*

Exponential growth can work for you if you invest your money rather than spend it all, or keep it under your mattress.

Suppose you were to invest \$100 at 7% interest right now. Next year you would have

$$\begin{aligned} & 100 + 100 \times 0.07 \\ &= 100(1 + 0.07) \\ &= 100(1.07) \\ &= 107 \text{ (dollars).} \end{aligned}$$

Two years from now you would have

$$\begin{aligned} & 100(1.07) + 100(1.07) \times 0.07 \\ &= 100(1.07)(1 + 0.07) \\ &= 100(1.07)(1.07) \\ &= 100(1.07)^2 \text{ (dollars).} \end{aligned}$$

After five years

$$100(1.07)^5 \text{ (dollars)}$$

"Arable land" means "land that will grow food".

MATTRESS SAVINGS YIELDS 0%

* $7\% = \frac{7}{100} = 0.07$

Money doesn't grow on trees, but it does grow at compound interest!

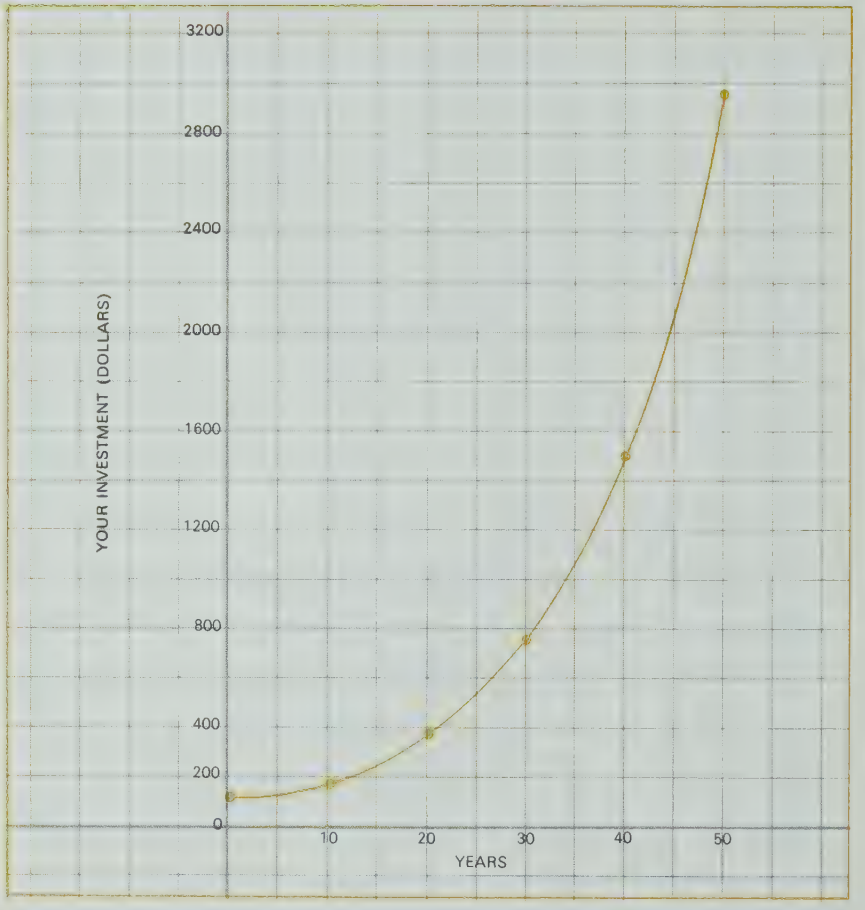
After ten years

$$100(1.07)^{10} \text{ (dollars)}$$

After twenty years

$$100(1.07)^{20} \text{ (dollars)}$$

The graph shows this exponential growth.



Exercise 25.1

Discussion

1. Discuss the implications of exponential growth as it relates to the quality of life in the future. Consider in your discussion
   (*a*) world population growth,
   (*b*) food production,
   (*c*) arable land requirements,
   (*d*) mineral supplies,
   (*e*) energy supplies,
   (*f*) water supply,
   (*g*) pollution problems.

Written Solutions

2. Draw a graph showing world population figures if the rate of increase was changed to
   (*a*) 0%,
   (*b*) 1%,
   (*c*) 3%.

3. Refer to the graph showing growth in savings. How long does it take for your money to
   (*a*) double?
   (*b*) become four times as much?
   (*c*) become eight times as much?
   (*d*) become sixteen times as much?

4. How would the situation described in Question 3 be changed if the interest was
   (*a*) 6%?
   (*b*) 10%?
   Draw a graph first.

3.(a) 10.2 years (b) 20.5 years
4.(a) 11.9 years, 23.8 years, 35.7 years, 47.6 years

25.2 The Exponential Relation

The following relations were developed to describe exponential growth.

(*a*) World population $= 3.30(1 + 0.021)^n \times 10^9$ where n is number of years from 1972.

(*b*) World arable land $= 1.47(1 + 0.021)^n \times 10^9$ ha where n is number of years from 1972.

$10^9 = 1$ billion

(*c*) Growth in savings

$$\text{Amount} = 100(1 + 0.07)^n$$

$100 invested at 7% for n years.

In every case where exponential growth is involved, you will find that a base is raised to an exponent as is the situation in each of the previous examples.

The general form of an exponential expression is a^x and by equating this to a second variable y we can form the general exponential relation

$$y = a^x$$

By substituting different values for the base, we can form many exponential relations.

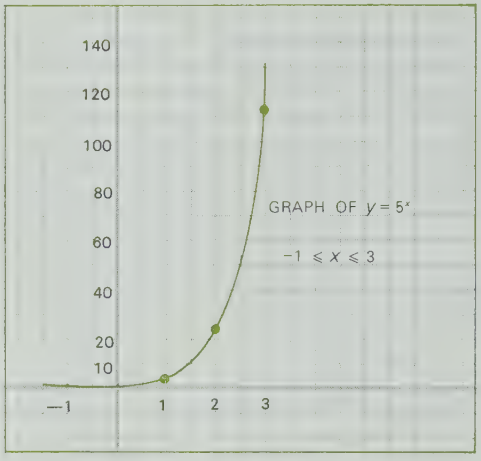
Example

Draw the graph of the exponential relation $y = 5^x$.

Solution

Make a table of values.

-1	0	1	2	3
$\frac{1}{5}$	1	5	25	125



Exercise 25.2

Discussion

1. Discuss the characteristics of an exponential curve using the graph of $y = 5^x$ as an example under the following headings.
   (*a*) The existence of a minimum value
   (*b*) The existence of a maximum value
   (*c*) Can the value for y be negative?
   (*d*) The value for y as x increases

Written Solutions

2. Draw the graphs of $y = 2^x$, $y = 3^x$, $y = 4^x$ on the same set of axes and discuss the effects of the change in base.

3. Draw the graphs of
   (*a*) $\{(x, y) | y = 4^x, -1 \leqslant x \leqslant 4, x \in \mathrm{R}\}$
   (*b*) $\{(x, y) | y = 7^x, -1 \leqslant x \leqslant 3, x \in \mathrm{R}\}$

4. Draw the graphs of
   (*a*) $\{(x, y) | y = 2^{-x}, -5 \leqslant x \leqslant 3, x \in \mathrm{R}\}$
   (*b*) $\{(x, y) | y = 3^{-x}, -4 \leqslant x \leqslant 1, x \in \mathrm{R}\}$
   (*c*) $\{(x, y) | y = 5^{-x}, -3 \leqslant x \leqslant 1, x \in \mathrm{R}\}$

5. The graphs of Question 4 are referred to as graphs of *exponential decay*. Explain why this is a very descriptive name for these graphs.

6. Do the graphs of Question 4
   (*a*) have a minimum value?
   (*b*) have a maximum value?
   (*c*) allow the value of y to be negative?
   (*d*) increase or decrease as x increases?

6.(b) No
(c) No

25.3 The Exponential Relation with Base 10

$y = 10^x$

This relation was considered briefly in Unit 1 when we were discussing standard notation.

You will recall that the use of standard notation allows us to consider any number, whether very large or very small, as a product of a number between 1 and 10 and some power of ten.

Examples

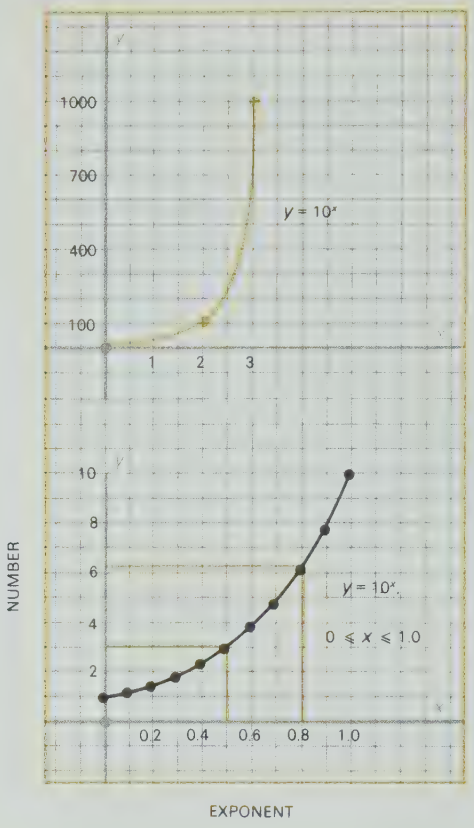
$$0.0436 = 4.36 \times 10^{-2}$$
decimal is 2 left of standard position

$$0.000\,000\,576 = 5.76 \times 10^{-7}$$
decimal is 7 left of standard position

$$1\,670\,000. = 1.67 \times 10^{6}$$
decimal is 6 right of standard position

Any exponent between 0 and 1 will give a number between 1 and 10.

From the graph,
$10^{0.5} \doteq 3.2$
and
$6.5 = 10^{0.82}$



This means that, instead of using the entire graph of $y = 10^x$, we need to consider only the part of it that gives numbers between 1 and 10.
This part would have exponents between 0 and 1.

Since $10^0 = 1$
and $10^1 = 10$
the section of the graph between 1 and 10 has been blown up to improve the accuracy of our work.

We can now complete the examples begun previously and express the three numbers as numbers in exponential form with base 10.

$$0.0436 = 4.36 \times 10^{-2}$$
$$\doteq 10^{0.65} \times 10^{-2} \quad \text{FROM THE GRAPH}$$
$$= 10^{-1.35}$$

$$0.000\,000\,576 = 5.76 \times 10^{-7}$$
$$= 10^{0.76} \times 10^{-7} \quad \text{FROM THE GRAPH}$$
$$= 10^{-6.24}$$

$$1\,670\,000 = 1.67 \times 10^{6}$$
$$= 10^{0.2} \times 10^{6}$$
$$= 10^{6.2}$$

This exponential relation may be used to make calculations that might not be so easily done otherwise.

Add the exponents.

NUMBER
↓
STANDARD FORM
↓
EXPONENTIAL FORM

Example

Calculate.

(*a*) $\sqrt{8.4}$

(*b*) $\sqrt[3]{47.6}$

Solution

(*a*) $\sqrt{8.4} = 8.4^{\frac{1}{2}}$

But $8.4 \doteq 10^{0.93}$

$\therefore\ 8.4^{\frac{1}{2}} \doteq (10^{0.93})^{\frac{1}{2}}$ ← MULTIPLY THE EXPONENTS

$\doteq 10^{0.47}$

$\doteq 2.9$

(*b*) $\sqrt[3]{47.6} = 47.6^{\frac{1}{3}}$

But $47.6 = 4.76 \times 10^{1}$

$\doteq 10^{0.68} \times 10^{1}$ ← ADD THE EXPONENTS

$= 10^{1.68}$

$\therefore\ 47.6^{\frac{1}{3}} \doteq (10^{1.68})^{\frac{1}{3}}$ ← MULTIPLY THE EXPONENTS

$= 10^{0.56}$

$\doteq 3.6$

Exercise 25.3

Use the graph of $y = 10^x$, $0 \leqslant x \leqslant 1$.

1. As accurately as you can, determine the numbers represented by the following.

(*a*) $10^{0.30}$	(*e*) $10^{0.9}$	(*i*) $10^{0.35}$
(*b*) $10^{0.20}$	(*f*) $10^{0.28}$	(*j*) $10^{0.25}$
(*c*) $10^{0.5}$	(*g*) $10^{0.45}$	(*k*) $10^{0.15}$
(*d*) $10^{0.7}$	(*h*) $10^{0.65}$	(*l*) $10^{0.92}$

2. As accurately as you can, determine the exponents for the numbers given when expressed with base 10.

(*a*) 2.3	(*e*) 4.5	(*i*) 4.2
(*b*) 5.6	(*f*) 1.6	(*j*) 2.9
(*c*) 7.2	(*g*) 8.7	(*k*) 5.0
(*d*) 9.5	(*h*) 6.8	(*l*) 8.1

1.(a) 2.00 **(b) 1.59** **(c) 3.16** **(d) 5.01**

2.(e) $10^{0.653}$ **(f) $10^{0.204}$** **(g) $10^{0.940}$** **(h) $10^{0.833}$**

Remember to write in standard notation first.

3. Express each of the following in exponential form with a base of 10.

(a) 16.9	(f) 980.0	(k) 6 900 000
(b) 740.0	(g) 25.0	(l) 92 000 000
(c) 160 000	(h) 0.000 67	(m) 0.000 709
(d) 0.0043	(i) 340 000	(n) 0.000 000 07
(e) 0.000 072	(j) 0.000 000 85	(o) 7 000 000

4. Calculate the following using the graph of $y = 10^x$.

(a) $\sqrt{8.2}$	(e) $\sqrt[4]{8.2}$	(i) $\sqrt[5]{45}$
(b) $\sqrt{7.5}$	(f) $\sqrt{15.3}$	(j) $\sqrt[3]{65}$
(c) $\sqrt{4.8}$	(g) $\sqrt{19.8}$	(k) $\sqrt[5]{95}$
(d) $\sqrt[3]{9.5}$	(h) $\sqrt[3]{12.5}$	(l) $\sqrt{35}$

3.(f) $10^{2.991}$ **(i)** $10^{5.532}$
(g) $10^{1.398}$ **(j)** $10^{-6.071}$
(h) $10^{-3.174}$

4.(i) 2.14 **(k) 2.49**
(j) 4.02 **(l) 5.92**

25.4 Tables of Exponents

Due to the inaccuracies that occur when using a graph as we did in the previous section, tables have been constructed that give the values of $y = 10^x$ for exponents between 0 and 1. These values have been calculated very accurately and can improve the accuracy of the work previously done.

Example 1

Determine the value of the following.

(a) $10^{0.723}$ (b) $10^{0.476}$ (c) $10^{2.238}$

Solution

(a) $10^{0.723} \doteq 5.284$
(b) $10^{0.476} \doteq 2.992$
(c) $10^{2.238} = 10^2 \times 10^{0.238}$
$\doteq 100 \times 1.730$
$= 173.0$

THE NUMBER IS ALWAYS BETWEEN 1 AND 10

EXPONENT 0.723
LINE
COLUMN

EXPONENTS				
	0	1	2	3
.87				7.464

Find 7.464 in the table on page 465 and read off the line and column 0.873

Example 2

Place in exponential form with base 10.

(a) 7.464 (b) 4.111 (c) 101 900

Solution

(a) $7.464 = 10^{0.873}$
(b) $4.111 = 10^{0.614}$
(c) $101\,900 = 1.019 \times 10^5$
$= 10^{0.008} \times 10^5$
$= 10^{5.008}$

Exercise 25.4

Use the table of values for $y = 10^x$.

1. Find the following numbers.

(a) $10^{0.286}$	(d) $10^{0.433}$	(g) $10^{0.430}$	(j) $10^{0.677}$
(b) $10^{0.573}$	(e) $10^{0.001}$	(h) $10^{0.950}$	(k) $10^{0.388}$
(c) $10^{0.821}$	(f) $10^{0.102}$	(i) $10^{0.205}$	(l) $10^{0.219}$

2. Determine the exponential form (base 10).

(a) 2.004	(f) 8.147	(k) 3.133	(p) 6.427
(b) 3.112	(g) 3.837	(l) 8.570	(q) 4.786
(c) 9.772	(h) 1.274	(m) 3.184	(r) 1.660
(d) 1.140	(i) 1.514	(n) 1.125	(s) 1.346
(e) 6.792	(j) 5.023	(o) 2.812	(t) 7.568

3. Find the following numbers.

(a) $10^{1.634}$	(i) $10^{6.191}$
(b) $10^{2.438}$	(j) $10^{0.549} \times 10^{-3}$
(c) $10^{4.333}$	(k) $10^{0.601} \times 10^{-2}$
(d) $10^{0.827} \times 10^{-2}$	(l) $10^{4.830}$
(e) $10^{0.628} \times 10^{-6}$	(m) $10^{6.109}$
(f) $10^{5.222}$	(n) $10^{3.483}$
(g) $10^{0.439} \times 10^{-5}$	(o) $10^{-5} \times 10^{0.391}$
(h) $10^{3.633}$	(p) $10^{-2} \times 10^{0.281}$

4. Determine the exponential form (base 10).

(a) 2.203	(g) 0.053 09	(m) 0.007 745
(b) 313.3	(h) 707.9	(n) 15.74
(c) 8730.0	(i) 54 200.0	(o) 2.692
(d) 0.098 40	(j) 0.006 383	(p) 28 770.0
(e) 0.4487	(k) 0.000 109 9	(q) 0.009 886
(f) 110.4	(l) 1256.0	(r) 0.000 173 4

5. Calculate the following using the table for $y = 10^x$.

(a) $\sqrt[3]{18.79}$	(i) $\sqrt[7]{6.714}$
(b) $\sqrt[3]{315.5}$	(j) $\sqrt[9]{319.9}$
(c) $\sqrt{9.772}$	(k) $\sqrt{2.265} \times \sqrt{1.629}$
(d) $\sqrt[4]{564.9}$	(l) $\sqrt[3]{154.9} \times \sqrt[7]{857.0}$
(e) $\sqrt[3]{2.280}$	(m) $\sqrt[5]{46.45} \times \sqrt[3]{389.9}$
(f) $\sqrt[5]{79.98}$	(n) $\sqrt[5]{444.6} \times \sqrt[10]{1406.0}$
(g) $\sqrt[7]{905.7}$	(o) $745.4^{\frac{1}{3}} \times 4.325^{\frac{1}{7}}$
(h) $\sqrt[5]{177.0}$	(p) $7.745^{\frac{1}{5}} \times 37.24^{\frac{1}{12}}$

$10^{1.634} = 10^1 \times 10^{0.634}$

TABLE

1.(a) 1.932 (c) 6.622
(b) 3.741 (d) 2.710

2.(a) $10^{0.302}$ (n) $10^{0.051}$
(b) $10^{0.493}$ (o) $10^{0.449}$
(c) $10^{0.990}$ (p) $10^{0.808}$
(h) $10^{0.105}$ (s) $10^{0.129}$
(i) $10^{0.180}$ (t) $10^{0.879}$
(j) $10^{0.701}$

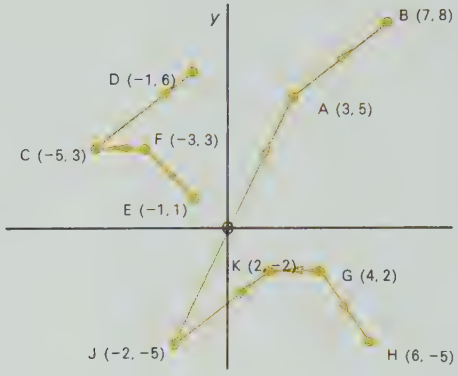
3.(a) 43.05
(b) 274.2
(g) 0.000 027 48
(h) 4295.0
(i) 1 552 000.0
(j) 0.003 540
(o) 0.00 024 60
(p) 0.019 10

4.(d) $10^{-1.007}$ (m) $10^{-2.111}$
(e) $10^{-0.348}$ (n) $10^{1.197}$
(i) $10^{4.734}$ (r) $10^{-3.761}$

5.(a) 2.659 (i) 1.312
(b) 6.808 (j) 1.898
(g) 2.644 (o) 11.18
(h) 2.816 (p) 2.035

Maintaining Basic Skills 7

1.



(*a*) Name the following vectors.
$\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$, $\overrightarrow{FC}$, $\overrightarrow{GH}$, $\overrightarrow{GK}$, $\overrightarrow{JK}$, $\overrightarrow{JA}$

(*b*) Which vectors are equal?

(*c*) Find the length of the following vectors.
$\overrightarrow{AB}$, $\overrightarrow{FC}$, $\overrightarrow{EF}$, $\overrightarrow{JA}$

2. Draw these position vectors, and find the length of each.

(*a*) $\vec{a} = (\overrightarrow{12, 5})$ (*c*) $\vec{c} = (\overrightarrow{-6, 2})$ (*e*) $\vec{e} = (\overrightarrow{0, 6})$

(*b*) $\vec{b} = (\overrightarrow{4, -3})$ (*d*) $\vec{d} = (\overrightarrow{-4, -7})$ (*f*) $\vec{f} = (\overrightarrow{-4, 0})$

3. Starting from the point (a, b), draw these vectors $(\overrightarrow{p, q})$.

Start (a, b)	Vector $(\overrightarrow{p, q})$
$(2, 4)$	$(\overrightarrow{3, -2})$
$(-3, 5)$	$(\overrightarrow{6, -3})$
$(2, -3)$	$(\overrightarrow{0, 8})$

Start (a, b)	Vector $(\overrightarrow{p, q})$
$(4, -2)$	$(\overrightarrow{-3, 7})$
$(-4, -6)$	$(\overrightarrow{-1, 8})$
$(-5, 2)$	$(\overrightarrow{7, 0})$

4. In Question 1, name the vector and give its ordered pair that is equivalent to the following vectors.

(*a*) $\overrightarrow{EF} + \overrightarrow{FC}$ (*d*) $\overrightarrow{JK} - \overrightarrow{GK}$

(*b*) $\overrightarrow{FC} + \overrightarrow{CD}$ (*e*) $\overrightarrow{EF} + \overrightarrow{FC} + \overrightarrow{CD}$

(*c*) $\overrightarrow{JA} + \overrightarrow{AB}$ (*f*) $\overrightarrow{JK} - \overrightarrow{GK} + \overrightarrow{GH}$

5. Express as a single vector.

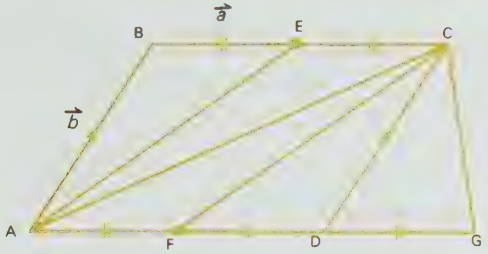
(*a*) $(\overrightarrow{3, 5}) - (\overrightarrow{-2, 6})$ (*d*) $5(\overrightarrow{-4, 1}) + (\overrightarrow{6, -5}) + 4(\overrightarrow{-2, -3})$

(*b*) $(\overrightarrow{8, 2}) + (\overrightarrow{5, 6})$ (*e*) $(\overrightarrow{7, -8}) - 2(\overrightarrow{4, -6}) - 3(\overrightarrow{-5, 2})$

(c) $3(\overrightarrow{2, -3}) + 2(\overrightarrow{5, 1})$ (*f*) $(\overrightarrow{0, 5}) - (\overrightarrow{6, 0}) - (\overrightarrow{-4, 3})$

1. (a) $\overrightarrow{AB} = (\overrightarrow{4, 3})$
$\overrightarrow{CD} = (\overrightarrow{4, 3})$
$\overrightarrow{EF} = (\overrightarrow{-2, 2})$
$\overrightarrow{FC} = (\overrightarrow{-2, 0})$
(b) $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{JK}$, $\overrightarrow{FC} = \overrightarrow{GK}$
2. (a) Length of $\vec{a} = 13$
(b) Length of $\vec{b} = 5$
(e) Length of $\vec{e} = 6$
4. (a) $\overrightarrow{EC} = (\overrightarrow{-4, 2})$
(b) $\overrightarrow{FD} = (\overrightarrow{2, 3})$
(c) $\overrightarrow{JB} = (\overrightarrow{9, 13})$
5. (a) $(\overrightarrow{5, -1})$
(b) $(\overrightarrow{13, 8})$
(c) $(\overrightarrow{16, -7})$

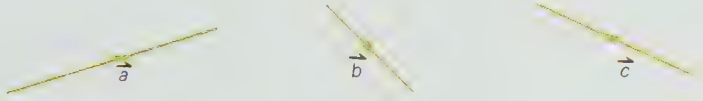
6.



ABCD is a parallelogram. $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{BE} = \vec{a}$. In terms of $\vec{a}$ or $\vec{b}$, or both, express these vectors.

(*a*) $\overrightarrow{AF}$ (*d*) $\overrightarrow{AC}$ (*g*) $\overrightarrow{CG}$
(*b*) $\overrightarrow{CD}$ (*e*) $\overrightarrow{CA}$ (*h*) $\overrightarrow{BD}$
(*c*) $\overrightarrow{AE}$ (*f*) $\overrightarrow{FC}$ (*i*) $\overrightarrow{BG}$

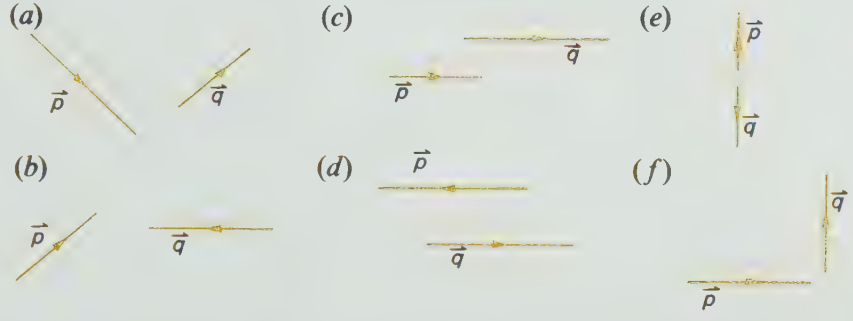
7.



Copy the vectors and draw the following.

(*a*) $\vec{a} + \vec{b}$ (*d*) $\vec{b} - \vec{c}$
(*b*) $\vec{a} + \vec{c}$ (*e*) $2\vec{a} - \vec{c}$
(*c*) $\vec{a} - \vec{b}$ (*f*) $2\vec{b} + 3\vec{c}$

8. Copy the vectors and draw the resultant of p and $\vec{q}$.



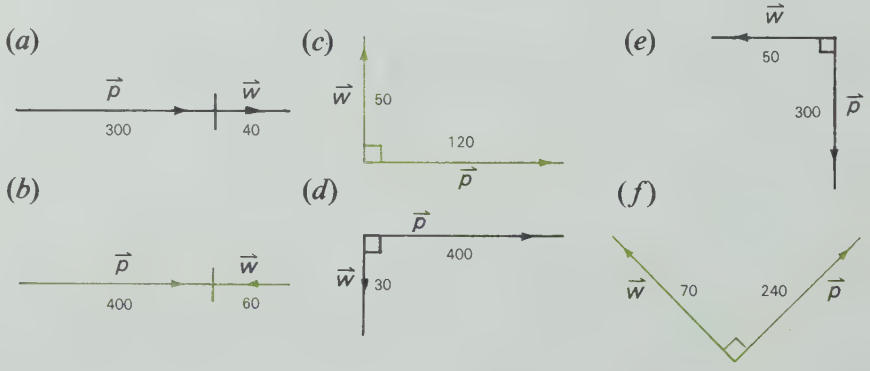
9. (*a*) Draw a diagram showing the position of a grasshopper which starts at A(0, 0) and jumps vector $\overrightarrow{(4, 1)}$ to B; from B he jumps $\overrightarrow{(3, -2)}$ to C; from C he jumps $\overrightarrow{(-5, -4)}$ to D.
(*b*) Express $\overrightarrow{AD}$ as an ordered pair.
(*c*) How far is the grasshopper from the starting point A?

6.(a) $\vec{a}$
(b) $-\vec{b}$
(c) $\vec{b} + \vec{a}$
(d) $\vec{b} + 2\vec{a}$
(e) $-2\vec{a} - \vec{b}$
9.(b) $\overrightarrow{AD} = \overrightarrow{(2, -5)}$
(c) $|AD| = \sqrt{29}$

10. In a golf game Jack tees off at point (3, 4). Units are 30 m The clubhouse is the origin. The ball follows vector $\overrightarrow{(2, -5)}$, then vector $\overrightarrow{(-1, 3)}$, then vector $\overrightarrow{(-1, -2)}$ into a sandtrap. Vector $\overrightarrow{(1, 0.5)}$ gets it out of the sandtrap. With vector $\overrightarrow{(4, -2)}$, it hits a tree. Then it bounces off the tree with $\overrightarrow{(1, 0.5)}$ into the creek.
    (*a*) Draw a diagram showing the paths the ball took.
    (*b*) Where was the tree? Where was the creek?
    (*c*) How far was the ball from the tee when it was in the creek?
    (*d*) How far, and in what direction relative to the x axis, did Jack walk to return to the clubhouse? (y axis is north)

11. An airplane starts from (6, 0). Its successive displacements are given by these vectors, where the unit is 100 km.
$$\overrightarrow{(2, 0)}, \overrightarrow{(0, 3)}, \overrightarrow{(-1, 2)}, \overrightarrow{(-3, -4)}, \overrightarrow{(-6, -7)}$$
    (*a*) Draw a diagram showing the flight path of the plane.
    (*b*) How far is the plane from the starting point?
    (*c*) What is the total distance travelled by the plane?

12. Calculate the magnitude of the resultant velocity in km/h of an airplane, where p is the air speed of the plane and w is the wind velocity (the direction to which the wind is blowing).



13. $\vec{b}$ is the velocity of a boat relative to the water, $\vec{c}$ is the velocity of the current in the river. Sketch a diagram and calculate the magnitude of the resultant velocity of the boat.
    (*a*) $\vec{b} = \overrightarrow{(20, 0)}, \vec{c} = \overrightarrow{(0, 4)}$
    (*b*) $\vec{b} = \overrightarrow{(0, 24)}, \vec{c} = \overrightarrow{(7, 0)}$
    (*c*) $\vec{b} = \overrightarrow{(4, 0)}, \vec{c} = \overrightarrow{(0, 3)}$
    (*d*) $\vec{b} = \overrightarrow{(0, 5)}, \vec{c} = \overrightarrow{(2, 0)}$

10. (b) The tree was at F(8, −1.5). The creek was at G(9, −1).
(c) 234 m

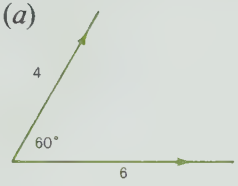
11. (b) 1000 km
(c) 2146 km

12. (a) 340 km/h
(b) 340 km/h
(c) 130 km/h

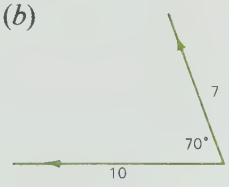
13. (a) $|\vec{r}| = 20.4$ units
(b) $|\vec{r}| = 25$ units

14. Calculate the resultant vector.

(*a*)



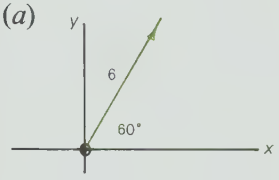
(*b*)



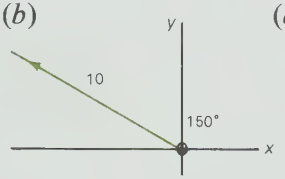
(*c*)

15. Change the vectors into the $(\overrightarrow{x, y})$ form.

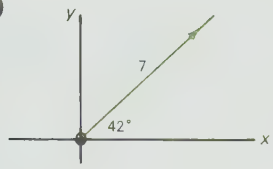
(*a*)



(*b*)

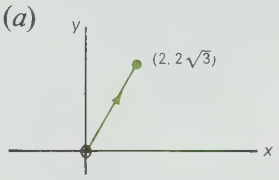


(*c*)

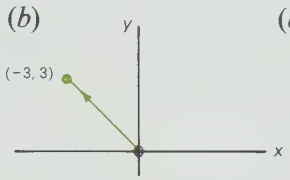


16. Express each vector in terms of r and θ.

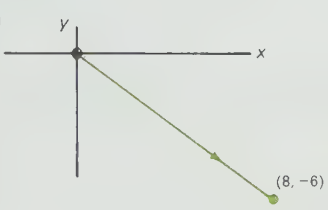
(*a*)



(*b*)



(*c*)



17. Express the vectors in the $(\overrightarrow{x, y})$ form.

(*a*) $4 \angle 45°$ (*c*) $50 \angle 300°$ (*e*) $10 \angle 75°$

(*b*) $80 \angle 120°$ (*d*) $12 \angle -30°$ (*f*) $1 \angle -40°$

18. Change each vector to the $r \angle \theta°$ form.

(*a*) $(\overrightarrow{4\sqrt{3}, 4})$ (*c*) $(\overrightarrow{-8, 8})$ (*e*) $(\overrightarrow{-3, 4})$

(*b*) $(\overrightarrow{6, -6\sqrt{3}})$ (*d*) $(\overrightarrow{5, -5})$ (*f*) $(\overrightarrow{5, -12})$

19. A tractor exerts a force of 900 N on a plow in a direction making a 25° angle with the horizontal. Find the component of the force along

(*a*) the x axis (pulling the plow forward)

(*b*) the y axis (lifting the plow up)

20. Calculate the following, using the table for $y = 10^x$.

(*a*) $\sqrt{28.32}$ (*d*) $\sqrt[3]{0.2570}$ (*g*) $\sqrt{7.824} \times \sqrt[3]{4.723}$

(*b*) $\sqrt{2.030}$ (*e*) $\sqrt[4]{29.82}$ (*h*) $\sqrt{0.08462} \times \sqrt[5]{10.26}$

(*c*) $\sqrt[3]{82.35}$ (*f*) $\sqrt[5]{172.6}$ (*i*) $\sqrt{6.452} \div \sqrt[4]{13.03}$

(a) 8.72
(b) 14.03
(c) 265

16. (a) $4 \angle 60°$

(b) $3\sqrt{2} \angle 135°$

17. (a) $(\overrightarrow{2\sqrt{2}, 2\sqrt{2}})$

(b) $(\overrightarrow{-40, 40\sqrt{3}})$

(c) $(\overrightarrow{25, -25\sqrt{3}})$

18. (a) $8 \angle 30°$

(b) $12 \angle 300°$

(c) $8\sqrt{2} \angle 135°$

19. (a) 816 N

20. (a) 5.322
(b) 1.424
(c) 4.351
(d) 0.6357
(e) 2.336

Summary of Units 18 to 25

	Do you understand . . . ?	*Can you . . . ?*
Simple and Compound Interest		
18.1 Simple Interest	the meaning of simple interest the difference between interest and interest rate	calculate simple interest state the formula for finding simple interest
18.2 Simple Interest Relationships	the relationship among interest, principal, interest rate, and time	solve for any one of the variables in the simple interest formula
18.3 Compound Interest	the difference between simple interest and compound interest how to calculate the interest rate per period how to calculate the number of interest periods	state the general formula for compounding principals
18.4 Compound Interest Tables	how to use the amount tables	evaluate $(1.04)^{10}$ and other like expressions
18.5 Problems Involving Amount	how to draw line diagrams for compounding amounts	calculate the interest per period and the number of interest periods
Applications of Percentage		
19.1 Cost Price and Selling Price	the meaning of prime cost and markup the difference between gross cost and prime cost the meaning of overhead and markup	state the relationship among selling price, prime cost, and markup state the relationship among gross cost, prime cost, and overhead use the formula markup = overhead + profit
19.2 Profit and Loss	how to calculate profit or rate of loss	state the relation among profit, selling price, and cost price
19.3 Trade Discounts	the purpose of trade discounts how to read the terms of a bill	calculate trade discounts
19.4 Commission	the meaning of sales commission	calculate wages based on (*a*) straight commission (*b*) salary plus commission (*c*) graduated commission

	Do you understand . . . ?	*Can you . . . ?*
Perimeter, Area, and Volume		
20.1 Perimeter	the meaning of perimeter	calculate the perimeter of a triangle, square, rectangle, circle, parallelogram, rhombus, trapezoid
20.2 Area	the meaning of area	calculate the area of the figures mentioned above
20.3 Surface Area of Right Prisms	the meaning of surface area of a triangular prism, circular prism, square prism, rectangular prism	calculate the surface area of each type using the formula $S = ph$
20.4 Volume of Right Prisms	the meaning of volume	calculate the volume of any right prism using the formula $V = Ah$
Taxes		
21.1 Federal Government	the major expenditures and sources of revenue for the Federal Government the present tax forms used by every taxpayer	read a circle graph construct a circle graph calculate your taxes or those of your family
21.2 Provincial Government	the major expenditures and sources of revenue for the provincial government the difference between direct and indirect taxes	list the main sources of revenue for the provinces
21.3 Municipal Government	the major responsibilities of your local municipality the major sources of revenue for your municipality	compare the role of the municipal government to that of the provincial and federal government

	Do you understand . . . ?	*Can you . . . ?*
Unit 22 **Ratio and Proportion**		
22.1 Ratio and Proportion Revisited	what is meant by equivalent ratios what constitutes a proportion what is meant by a mean proportional	identify the first, second, third, and fourth proportional in a proportion
22.2 Proportions in Geometry	what is necessary to make triangles similar which ratios are equal in similar triangles and how you pick them the symbols $=$, $\sim$, $\cong$	identify a pair of similar triangles calculate to find missing dimensions in similar triangles
22.3 Gear Drives	the proportion used in calculations with gear drives, belt drives, and chain drives what is meant by an idler, a simple train, a compound train	solve proportions with gear, belt, and chain drives
22.4 Belt Drives	the purpose of open and cross-type belt drives	
Unit 23 **Mathematics and Chance**		
23.1 Probability	the meaning of probability how to construct a tree diagram the meaning of odds	calculate the probability of an event happening . . . or not happening explain why we use tree diagrams calculate the odds in favor of an event
23.2 The Fundamental Counting Principle	the meaning of the fundamental counting principle	solve basic "how many ways" problems
23.3 Permutations	the meaning of a permutation the meaning of factorial notation	calculate a "number of arrangements"
23.4 Combinations	the meaning of a combination	calculate a "number of selections or choices" distinguish between "arrangement" and "selection" problems

	Do you understand . . . ?	*Can you . . . ?*
Unit 24		
Vector and Analytical Trigonometry		
24.1 Vector Representation	what is required to describe a vector what is meant by opposite vectors	identify equal vectors
24.2 Vectors as Ordered Pairs	when a vector is called a position vector describe opposite vectors using ordered pairs	describe a vector using ordered pairs
24.3 Addition of Vectors	how to add vectors	find the resultant vector
24.4 Multiplication of a Vector by a Scalar	the difference between a vector and a scalar	find the resultant vector when a vector is multiplied by a scalar
24.5 Subtraction of Vectors	how to subtract vectors described by ordered pairs	
24.6 Vectors and Trigonometry	how to describe a vector by means of its magnitude and angle of rotation from a reference axis how to convert a vector described by magnitude and angle of rotation to a vector described by an ordered pair	interpret $r\underline{/\theta}$ state the relationship among r, x and y, and r, $\sin\theta$ and $\cos\theta$
24.7 Solving Problems with Vectors	the meaning of a resultant of a system of vectors how to use the Sine Law and the Cosine Law in solving vector problems	find the direction of a resultant vector
Unit 25		
The Exponential Relation		
25.1 Exponential Growth	the significance of exponential growth in the world around you	compare the growth of savings at various percentages
25.2 The Exponential Relation	the characteristics of the general exponential relation	graph exponential relations with various bases
25.3 The Exponential Relation with Base 10	how the exponential relation with base 10 ties in with the number system of base 10	use the exponential relation with base 10 to make calculations
25.4 Tables of Exponents	the relationship of the tables of exponents to the exponential relation	use the tables to make calculations

Review Exercise, Units 18 to 25

1. Solve for the variables.

(*a*) $\frac{12}{x} = \frac{3}{7}$ (*c*) $\frac{15}{w} = \frac{24}{5}$ (*e*) $\frac{35}{6y} = \frac{14}{3}$

(*b*) $\frac{6}{5} = \frac{4t}{9}$ (*d*) $\frac{4}{3} = \frac{20}{r}$ (*f*) $\frac{1}{2} = \frac{6}{5x}$

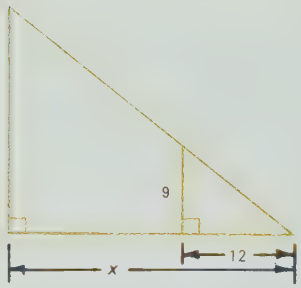
2. Find the mean proportional between the given numbers.

(*a*) 3, 12 (*c*) $8a^3$, $18a$ (*e*) 5, 8

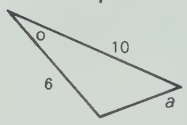
(*b*) 32, 8 (*d*) $8\sqrt{2}$, $\sqrt{2}$ (*f*) $2\sqrt{6}$, $3\sqrt{6}$

3. Find the value of each variable.

(*a*)

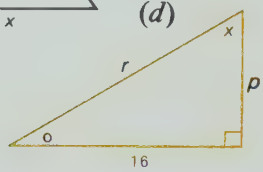


(*b*)

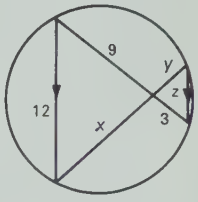


(*c*)

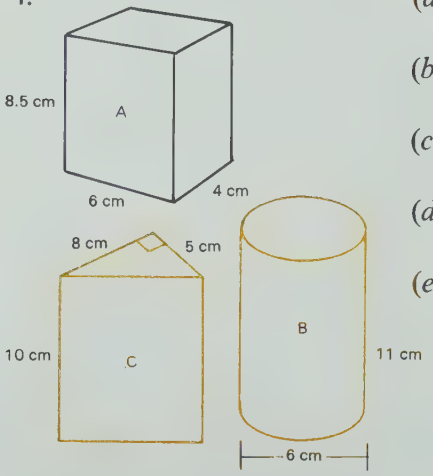
(*d*)



(*e*)



4.



(*a*) Find the area of the top of each container.

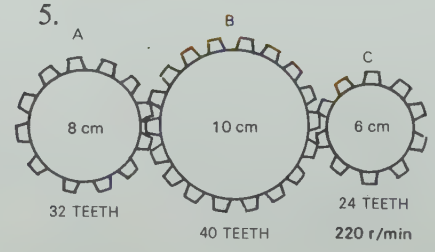
(*b*) Find the total surface area of each.

(*c*) Find the volume of each container.

(*d*) What is the ratio of the volume of A to B? Of A to C?

(*e*) A metal strap is fixed around the middle of each container before shipping. How much strapping is needed for each? (Allow 8 cm overlap.)

5.



(*a*) Calculate the area of each wheel.

(*b*) Find the speed of wheel B, and of wheel A.

1.(a) 28
(b) $2\frac{7}{10}$
(c) $\frac{25}{8}$

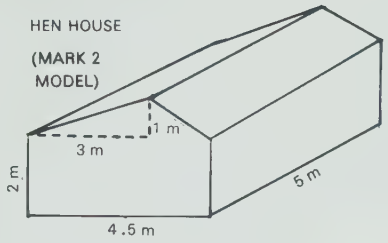
2.(a) 6
(b) 16
(c) $12a^2$

3.(a) $x = 28$
(b) $a = 4\frac{2}{3}$, $b = 9$

4.(a) 24 cm², 28.26 cm², 20 cm²
(b) 218 cm², 264 cm², 264 cm²

5.(b) 132 r/min, 165 r/min

6.



Calculate the following.

(*a*) the slope of each part of the roof
(*b*) the area of the roof
(*c*) the area of each end
(*d*) the volume
(*e*) the number of hens it could hold, if one hen needs 3750 cm^2 floor space

7. Complete the table using simple interest.

Principal	*Rate*	*Time*	*Interest*
\$2000	9%	5 years	
\$ 40		2 months	\$1.20
\$ 70	24%	73 days	

8. Use compound interest tables to find the amount.

Principal	*Rate*	*Time*	*Compounded*	*Amount*
\$ 50	10%	25 years	annually	
\$3000	12%	15 years	semi-annually	
\$ 400	16%	$2\frac{1}{2}$ years	quarterly	
\$ 150	18%	13 months	monthly	

9. Complete the following table.

Selling Price	*Prime Cost*	*Overhead*	*Markup*	*Profit*
\$ 120	\$ 70	20%		
\$3000	\$1400			\$600
\$ 40	\$ 25	\$8		

10. Complete the following table.

List Price	*Trade Discount*	*Amount*
\$ 35	20%, 10%	
\$480	$33\frac{1}{3}$%, 5%	

11. Simplify the vectors.

(*a*) $2\overrightarrow{(6, 1)} + 3\overrightarrow{(-4, 2)}$
(*b*) $\overrightarrow{(8, -4)} - \overrightarrow{(9, 3)}$
(*c*) $2\overrightarrow{(-5, 2)} - \overrightarrow{(-3, 4)}$

6.(b) **25 m^2**
(c) **11.25 m^2**
(d) **56.25 m^3**

8.

Principal	*Amount*
\$50	**\$541.74**
\$3000	**\$17 230.47**

9.

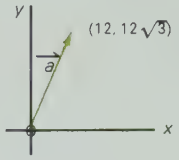
Selling Price	*Markup*
\$120	**\$50**
\$3000	**\$1600**

10.

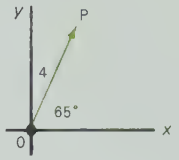
List Price	*Amount*
\$35	**\$25.20**

11(a) $\overrightarrow{(0, 8)}$

12. Express $\overrightarrow{a}$ as $(r \angle \theta°)$.



13. Express $\overrightarrow{OP}$ as $(\overrightarrow{x, y})$



14. Chester Field is the proprietor of the Home Suite Home Furniture Store. His sales for a particular month were \$36 000 and his gross cost was \$32 000.
   (*a*) What is his profit?
   (*b*) What is his percentage profit?

15. On March 20th, Chester invested \$20 000 with a trust company at 8% compounded quarterly, so that he would have enough money to buy new stock at Christmas. On December 20th, how much money would he have to his credit?

16. Mr. Woody Teak is one of the senior salesmen at the Home Suite Home Furniture Store. In addition to his salary of \$400 a month, he receives a 5% commission on all sales up to \$2000 and 6% on sales over \$2000. During July, Woody sold 7 living room suites at \$450 each, 5 bedroom suites at \$420 each, 3 tables at \$150 each, and 4 rugs at \$220 each. What were his total earnings for the month?

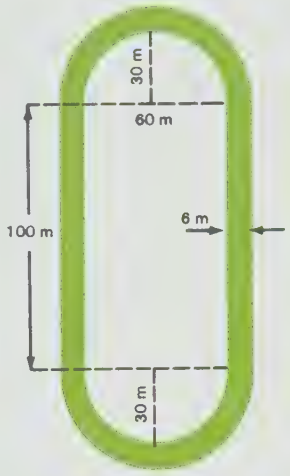
17. Woody is a married man with two children under 16 years old. His total earnings for the year amounted to \$11 230 with an added income from bonds of \$165. If he has already paid \$820 in income tax deductions and given \$96 to charity, calculate how much tax Woody still has to pay.

18. This year, Woody is planning on having a new house built. Five years ago today, he invested \$2000, a year later another \$2000, and three years ago still another \$2000. Each sum earns 9% interest compounded annually. How much money will he have now for a down payment?

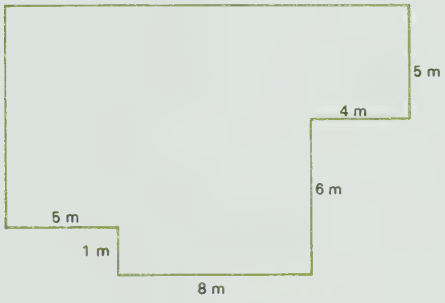
19. A young, single girl, Ima Lone, has been working at the store for only a few months. Her income for this period was \$2600 from which she has paid \$480 in income tax deductions. Will she have to pay more or will she get a refund? How much?

13. $\overrightarrow{(1.6904, 3.6252)}$
14(a) \$4000
16. \$774.80
18. \$8490.46

20. Woody Teak's son Russ, is practising for the school track and field events. The cinder track has semicircular ends and straight stretches between. The dimensions are shown.
    (*a*) What distance would Russ run in 1 lap if he kept to the inside edge of the track?
    (*b*) How far would he run if he followed the outside edge?
    (*c*) What distance would a runner save in 1 lap if he could keep the inside track?
    (*d*) Calculate the area of the grass playing field inside the track.
    (*e*) Find the area that is covered with cinders.
    (*f*) What is the ratio of the grass area to the cinders area?
    (*g*) If the cinders on the track are 15 cm deep, find the volume of the cinders.



21. Consider the Teaks' house plan shown in the diagram.
    (*a*) Calculate the floor area.
    (*b*) At \$320/m^2, how much would the house cost?
    (*c*) How much eavestroughing would be required for the whole house? (Neglect the width of the eavestroughing.)



22. The Teaks wish to install an air conditioner in a bedroom that is 4.4 m long, 3.6 m wide, and 2.5 m high.
    (*a*) What is the volume of the room?
    (*b*) If air conditioners are rated at 12 000 kJ for 20 m^3, what size will they need (in kJ)?

1 kJ = 1 kilojoule

23. Woody Teak's daughter, Polly, reaches into the high cupboard for a tin of soup. There are 3 tins of tomato soup, 2 of vegetable, 4 of pea, and 1 of onion. What is the probability she chooses the following.
    (*a*) tomato soup
    (*b*) pea soup
    (*c*) vegetable soup
    (*d*) not onion soup

24. Floor tiles for the kitchen come in grey, beige, and green. Arborite countertops are available in white, gold, and green. Woody has no idea what his wife wants so he picks colors at random. Anne really wants white arborite with beige or green tiles. What is the probability she gets what she wants?

20(a) 388.5 m
(b) 426.2 m
(c) 37.7 m
21(a) 158 m^2
(b) \$50 560
22(b) 23 760 kJ
23 (a) $\frac{3}{10}$
(b) $\frac{2}{5}$

TABLE OF TRIGONOMETRIC VALUES

θ	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
0	0.0000	1.0000	0.0000		1.0000	
1	0.0175	0.9998	0.0175	57.299	1.0002	57.290
2	0.0349	0.9994	0.0349	28.654	1.0006	28.636
3	0.0523	0.9986	0.0524	19.107	1.0014	19.081
4	0.0698	0.9976	0.0699	14.336	1.0024	14.301
5	0.0872	0.9962	0.0875	11.474	1.0038	11.4301
6	0.1045	0.9945	0.1051	9.5668	1.0055	9.5144
7	0.1219	0.9925	0.1228	8.2055	1.0075	8.1443
8	0.1392	0.9903	0.1405	7.1853	1.0098	7.1154
9	0.1564	0.9877	0.1584	6.3925	1.0125	6.3138
10	0.1736	0.9848	0.1763	5.7588	1.0154	5.6713
11	0.1908	0.9816	0.1944	5.2408	1.0187	5.1446
12	0.2079	0.9781	0.2126	4.8097	1.0223	4.7046
13	0.2250	0.9744	0.2309	4.4454	1.0263	4.3315
14	0.2419	0.9703	0.2493	4.1336	1.0306	4.0108
15	0.2588	0.9659	0.2679	3.8637	1.0353	3.7321
16	0.2756	0.9613	0.2867	3.6280	1.0403	3.4874
17	0.2924	0.9563	0.3057	3.4203	1.0457	3.2709
18	0.3090	0.9511	0.3249	3.2361	1.0515	3.0777
19	0.3256	0.9455	0.3443	3.0716	1.0576	2.9042
20	0.3420	0.9397	0.3640	2.9238	1.0642	2.7475
21	0.3584	0.9336	0.3839	2.7904	1.0711	2.6051
22	0.3746	0.9272	0.4040	2.6695	1.0785	2.4751
23	0.3907	0.9205	0.4245	2.5593	1.0864	2.3559
24	0.4067	0.9135	0.4452	2.4586	1.0946	2.2460
25	0.4226	0.9063	0.4663	2.3662	1.1034	2.1445
26	0.4384	0.8988	0.4877	2.2812	1.1126	2.0503
27	0.4540	0.8910	0.5095	2.2027	1.1223	1.9626
28	0.4695	0.8829	0.5317	2.1301	1.1326	1.8807
29	0.4848	0.8746	0.5543	2.0627	1.1434	1.8041
30	0.5000	0.8660	0.5774	2.0000	1.1547	1.7321
31	0.5150	0.8572	0.6009	1.9416	1.1666	1.6643
32	0.5299	0.8480	0.6249	1.8871	1.1792	1.6003
33	0.5446	0.8387	0.6494	1.8361	1.1924	1.5399
34	0.5592	0.8290	0.6745	1.7883	1.2062	1.4826
35	0.5736	0.8192	0.7002	1.7434	1.2208	1.4281
36	0.5878	0.8090	0.7265	1.7013	1.2361	1.3764
37	0.6018	0.7986	0.7536	1.6616	1.2521	1.3270
38	0.6157	0.7880	0.7813	1.6243	1.2690	1.2799
39	0.6293	0.7771	0.8098	1.5890	1.2868	1.2349
40	0.6428	0.7660	0.8391	1.5557	1.3054	1.1918
41	0.6561	0.7547	0.8693	1.5243	1.3250	1.1504
42	0.6691	0.7431	0.9004	1.4945	1.3456	1.1106
43	0.6820	0.7314	0.9325	1.4663	1.3673	1.0724
44	0.6947	0.7193	0.9657	1.4396	1.3902	1.0355

TABLE OF TRIGONOMETRIC VALUES

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
45	0.7071	0.7071	1.0000	1.4142	1.4142	1.0000
46	0.7193	0.6947	1.0355	1.3902	1.4396	0.9657
47	0.7314	0.6820	1.0724	1.3673	1.4663	0.9325
48	0.7431	0.6691	1.1106	1.3456	1.4945	0.9004
49	0.7547	0.6561	1.1504	1.3250	1.5243	0.8693
50	0.7660	0.6428	1.1918	1.3054	1.5557	0.8391
51	0.7771	0.6293	1.2349	1.2868	1.5890	0.8098
52	0.7880	0.6157	1.2799	1.2690	1.6243	0.7813
53	0.7986	0.6018	1.3270	1.2521	1.6616	0.7536
54	0.8090	0.5878	1.3764	1.2361	1.7013	0.7265
55	0.8192	0.5736	1.4281	1.2208	1.7434	0.7002
56	0.8290	0.5592	1.4826	1.2062	1.7883	0.6745
57	0.8387	0.5446	1.5399	1.1924	1.8361	0.6494
58	0.8480	0.5299	1.6003	1.1792	1.8871	0.6249
59	0.8572	0.5150	1.6643	1.1666	1.9416	0.6009
60	0.8660	0.5000	1.7321	1.1547	2.0000	0.5774
61	0.8746	0.4848	1.8041	1.1434	2.0627	0.5543
62	0.8829	0.4695	1.8807	1.1326	2.1301	0.5317
63	0.8910	0.4540	1.9626	1.1223	2.2027	0.5095
64	0.8988	0.4384	2.0503	1.1126	2.2812	0.4877
65	0.9063	0.4226	2.1445	1.1034	2.3662	0.4663
66	0.9135	0.4067	2.2460	1.0946	2.4586	0.4452
67	0.9205	0.3907	2.3559	1.0864	2.5593	0.4245
68	0.9272	0.3746	2.4751	1.0785	2.6695	0.4040
69	0.9336	0.3584	2.6051	1.0711	2.7904	0.3839
70	0.9397	0.3420	2.7475	1.0642	2.9238	0.3640
71	0.9455	0.3256	2.9042	1.0576	3.0716	0.3443
72	0.9511	0.3090	3.0777	1.0515	3.2361	0.3249
73	0.9563	0.2924	3.2709	1.0457	3.4203	0.3057
74	0.9613	0.2756	3.4874	1.0403	3.6280	0.2867
75	0.9659	0.2588	3.7321	1.0353	3.8637	0.2679
76	0.9703	0.2419	4.0108	1.0306	4.1336	0.2493
77	0.9744	0.2250	4.3315	1.0263	4.4454	0.2309
78	0.9781	0.2079	4.7046	1.0223	4.8097	0.2126
79	0.9816	0.1908	5.1446	1.0187	5.2408	0.1944
80	0.9848	0.1736	5.6713	1.0154	5.7588	0.1763
81	0.9877	0.1564	6.3138	1.0125	6.3925	0.1584
82	0.9903	0.1392	7.1154	1.0098	7.1853	0.1405
83	0.9925	0.1219	8.1443	1.0075	8.2055	0.1228
84	0.9945	0.1045	9.5144	1.0055	9.5668	0.1051
85	0.9962	0.0872	11.4301	1.0038	11.474	0.0875
86	0.9976	0.0698	14.301	1.0024	14.336	0.0699
87	0.9986	0.0523	19.081	1.0014	19.107	0.0524
88	0.9994	0.0349	28.636	1.0006	28.654	0.0349
89	0.9998	0.0175	57.290	1.0002	57.299	0.0175
90	1.0000	0.0000		1.0000		0.0000

EXPONENTIAL TABLE: $y = 10^x$ (ANTILOGARITHMS)

	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
.00	1.000	1.002	1.005	1.007	1.009	1.012	1.014	1.016	1.019	1.021	0	0	1	1	1	1	2	2	2
.01	1.023	1.026	1.028	1.030	1.033	1.035	1.038	1.040	1.042	1.045	0	0	1	1	1	1	2	2	2
.02	1.047	1.050	1.052	1.054	1.057	1.059	1.062	1.064	1.067	1.069	0	0	1	1	1	1	2	2	2
.03	1.072	1.074	1.076	1.079	1.081	1.084	1.086	1.089	1.091	1.094	0	0	1	1	1	1	2	2	2
.04	1.096	1.099	1.102	1.104	1.107	1.109	1.112	1.114	1.117	1.119	0	1	1	1	1	2	2	2	2
.05	1.122	1.125	1.127	1.130	1.132	1.135	1.138	1.140	1.143	1.146	0	1	1	1	1	2	2	2	2
.06	1.148	1.151	1.153	1.156	1.159	1.161	1.164	1.167	1.169	1.172	0	1	1	1	1	2	2	2	2
.07	1.175	1.178	1.180	1.183	1.186	1.189	1.191	1.194	1.197	1.199	0	1	1	1	1	2	2	2	2
.08	1.202	1.205	1.208	1.211	1.213	1.216	1.219	1.222	1.225	1.227	0	1	1	1	1	2	2	2	3
.09	1.230	1.233	1.236	1.239	1.242	1.245	1.247	1.250	1.253	1.256	0	1	1	1	1	2	2	2	3
.10	1.259	1.262	1.265	1.268	1.271	1.274	1.276	1.279	1.282	1.285	0	1	1	1	1	2	2	2	3
.11	1.288	1.291	1.294	1.297	1.300	1.303	1.306	1.309	1.312	1.315	0	1	1	1	2	2	2	2	3
.12	1.318	1.321	1.324	1.327	1.330	1.334	1.337	1.340	1.343	1.346	0	1	1	1	2	2	2	2	3
.13	1.349	1.352	1.355	1.358	1.361	1.365	1.368	1.371	1.374	1.377	0	1	1	1	2	2	2	3	3
.14	1.380	1.384	1.387	1.390	1.393	1.396	1.400	1.403	1.406	1.409	0	1	1	1	2	2	2	3	3
.15	1.413	1.416	1.419	1.422	1.426	1.429	1.432	1.435	1.439	1.442	0	1	1	1	2	2	2	3	3
.16	1.445	1.449	1.452	1.455	1.459	1.462	1.466	1.469	1.472	1.476	0	1	1	1	2	2	2	3	3
.17	1.479	1.483	1.486	1.489	1.493	1.496	1.500	1.503	1.507	1.510	0	1	1	1	2	2	2	3	3
.18	1.514	1.517	1.521	1.524	1.528	1.531	1.535	1.538	1.542	1.545	0	1	1	1	2	2	2	3	3
.19	1.549	1.552	1.556	1.560	1.563	1.567	1.570	1.574	1.578	1.581	0	1	1	1	2	2	3	3	3
.20	1.585	1.589	1.592	1.596	1.600	1.603	1.607	1.611	1.614	1.618	0	1	1	1	2	2	3	3	3
.21	1.622	1.626	1.629	1.633	1.637	1.641	1.644	1.648	1.652	1.656	0	1	1	2	2	2	3	3	3
.22	1.660	1.663	1.667	1.671	1.675	1.679	1.683	1.687	1.690	1.694	0	1	1	2	2	2	3	3	3
.23	1.698	1.702	1.706	1.710	1.714	1.718	1.722	1.726	1.730	1.734	0	1	1	2	2	2	3	3	4
.24	1.738	1.742	1.746	1.750	1.754	1.758	1.762	1.766	1.770	1.774	0	1	1	2	2	2	3	3	4
.25	1.778	1.782	1.786	1.791	1.795	1.799	1.803	1.807	1.811	1.816	0	1	1	2	2	2	3	3	4
.26	1.820	1.824	1.828	1.832	1.837	1.841	1.845	1.849	1.854	1.858	0	1	1	2	2	3	3	3	4
.27	1.862	1.866	1.871	1.875	1.879	1.884	1.888	1.892	1.897	1.901	0	1	1	2	2	3	3	3	4
.28	1.905	1.910	1.914	1.919	1.923	1.928	1.932	1.936	1.941	1.945	0	1	1	2	2	3	3	4	4
.29	1.950	1.954	1.959	1.963	1.968	1.972	1.977	1.982	1.986	1.991	0	1	1	2	2	3	3	4	4
.30	1.995	2.000	2.004	2.009	2.014	2.018	2.023	2.028	2.032	2.037	0	1	1	2	2	3	3	4	4
.31	2.042	2.046	2.051	2.056	2.061	2.065	2.070	2.075	2.080	2.084	0	1	1	2	2	3	3	4	4
.32	2.089	2.094	2.099	2.104	2.109	2.113	2.118	2.123	2.128	2.133	0	1	1	2	2	3	3	4	4
.33	2.138	2.143	2.148	2.153	2.158	2.163	2.168	2.173	2.178	2.183	0	1	1	2	2	3	3	4	4
.34	2.188	2.193	2.198	2.203	2.208	2.213	2.218	2.223	2.228	2.234	1	1	2	2	3	3	4	4	5
.35	2.239	2.244	2.249	2.254	2.259	2.265	2.270	2.275	2.280	2.286	1	1	2	2	3	3	4	4	5
.36	2.291	2.296	2.301	2.307	2.312	2.317	2.323	2.328	2.333	2.339	1	1	2	2	3	3	4	4	5
.37	2.344	2.350	2.355	2.360	2.366	2.371	2.377	2.382	2.388	2.393	1	1	2	2	3	3	4	4	5
.38	2.399	2.404	2.410	2.415	2.421	2.427	2.432	2.438	2.443	2.449	1	1	2	2	3	3	4	4	5
.39	2.455	2.460	2.466	2.472	2.477	2.483	2.489	2.495	2.500	2.506	1	1	2	2	3	3	4	5	5
.40	2.512	2.518	2.523	2.529	2.535	2.541	2.547	2.553	2.559	2.564	1	1	2	2	3	4	4	5	5
.41	2.570	2.576	2.582	2.588	2.594	2.600	2.606	2.612	2.618	2.624	1	1	2	2	3	4	4	5	5
.42	2.630	2.636	2.642	2.649	2.655	2.661	2.667	2.673	2.679	2.685	1	1	2	2	3	4	4	5	6
.43	2.692	2.698	2.704	2.710	2.716	2.723	2.729	2.735	2.742	2.748	1	1	2	3	3	4	4	5	6
.44	2.754	2.761	2.767	2.773	2.780	2.786	2.793	2.799	2.805	2.812	1	1	2	3	3	4	4	5	6
.45	2.818	2.825	2.831	2.838	2.844	2.851	2.858	2.864	2.871	2.877	1	1	2	3	3	4	5	5	6
.46	2.884	2.891	2.897	2.904	2.911	2.917	2.924	2.931	2.938	2.944	1	1	2	3	3	4	5	5	6
.47	2.951	2.958	2.965	2.972	2.979	2.985	2.992	2.999	3.006	3.013	1	1	2	3	3	4	5	5	6
.48	3.020	3.027	3.034	3.041	3.048	3.055	3.062	3.069	3.076	3.083	1	1	2	3	4	4	5	6	6
.49	3.090	3.097	3.105	3.112	3.119	3.126	3.133	3.141	3.148	3.155	1	1	2	3	4	4	5	6	6

EXPONENTIAL TABLE: $y = 10^x$ (ANTILOGARITHMS)

	0	1	2	3	4	5	6	7	8	9	Differences		
											1 2 3	4 5 6	7 8 9
.50	3.162	3.170	3.177	3.184	3.192	3.199	3.206	3.214	3.221	3.228	1 1 2	3 4 4	5 6 7
.51	3.236	3.243	3.251	3.258	3.266	3.273	3.281	3.289	3.296	3.304	1 2 2	3 4 5	5 6 7
.52	3.311	3.319	3.327	3.334	3.342	3.350	3.357	3.365	3.373	3.381	1 2 2	3 4 5	5 6 7
.53	3.388	3.396	3.404	3.412	3.420	3.428	3.436	3.443	3.451	3.459	1 2 2	3 4 5	6 6 7
.54	3.467	3.475	3.483	3.491	3.499	3.508	3.516	3.524	3.532	3.540	1 2 2	3 4 5	6 6 7
.55	3.548	3.556	3.565	3.573	3.581	3.589	3.597	3.606	3.613	3.622	1 2 2	3 4 5	6 7 7
.56	3.631	3.639	3.648	3.656	3.664	3.673	3.681	3.690	3.698	3.707	1 2 3	3 4 5	6 7 8
.57	3.715	3.724	3.733	3.741	3.750	3.758	3.767	3.776	3.784	3.793	1 2 3	3 4 5	6 7 8
.58	3.802	3.811	3.819	3.828	3.837	3.846	3.855	3.864	3.873	3.882	1 2 3	4 4 5	6 7 8
.59	3.890	3.899	3.908	3.917	3.926	3.936	3.945	3.954	3.963	3.972	1 2 3	4 5 5	6 7 8
.60	3.981	3.990	3.999	4.009	4.018	4.027	4.036	4.046	4.055	4.064	1 2 3	4 5 6	6 7 8
.61	4.074	4.083	4.093	4.102	4.111	4.121	4.130	4.140	4.150	4.159	1 2 3	4 5 6	7 8 9
.62	4.169	4.178	4.188	4.198	4.207	4.217	4.227	4.236	4.246	4.256	1 2 3	4 5 6	7 8 9
.63	4.266	4.276	4.285	4.295	4.305	4.315	4.325	4.335	4.345	4.355	1 2 3	4 5 6	7 8 9
.64	4.365	4.375	4.385	4.395	4.406	4.416	4.426	4.436	4.446	4.457	1 2 3	4 5 6	7 8 9
.65	4.467	4.477	4.487	4.498	4.508	4.519	4.529	4.539	4.550	4.560	1 2 3	4 5 6	7 8 9
.66	4.571	4.581	4.592	4.603	4.613	4.624	4.634	4.645	4.656	4.667	1 2 3	4 5 6	7 9 10
.67	4.677	4.688	4.699	4.710	4.721	4.732	4.742	4.753	4.764	4.775	1 2 3	4 5 7	8 9 10
.68	4.786	4.797	4.808	4.819	4.831	4.842	4.853	4.864	4.875	4.887	1 2 3	4 6 7	8 9 10
.69	4.898	4.909	4.920	4.932	4.943	4.955	4.966	4.977	4.989	5.000	1 2 3	5 6 7	8 9 10
.70	5.012	5.023	5.035	5.047	5.058	5.070	5.082	5.093	5.105	5.117	1 2 4	5 6 7	8 9 11
.71	5.129	5.140	5.152	5.164	5.176	5.188	5.200	5.212	5.224	5.236	1 2 4	5 6 7	8 10 11
.72	5.248	5.260	5.272	5.284	5.297	5.309	5.321	5.333	5.346	5.358	1 2 4	5 6 7	9 10 11
.73	5.370	5.383	5.395	5.408	5.420	5.433	5.445	5.458	5.470	5.483	1 3 4	5 6 8	9 10 11
.74	5.495	5.508	5.521	5.534	5.546	5.559	5.572	5.585	5.598	5.610	1 3 4	5 6 8	9 10 12
.75	5.623	5.636	5.649	5.662	5.675	5.689	5.702	5.715	5.728	5.741	1 3 4	5 7 8	9 10 12
.76	5.754	5.768	5.781	5.794	5.808	5.821	5.834	5.848	5.861	5.875	1 3 4	5 7 8	9 11 12
.77	5.888	5.902	5.916	5.929	5.943	5.957	5.970	5.984	5.998	6.012	1 3 4	5 7 8	10 11 12
.78	6.026	6.039	6.053	6.067	6.081	6.095	6.109	6.124	6.138	6.152	1 3 4	6 7 8	10 11 13
.79	6.166	6.180	6.194	6.209	6.223	6.236	6.252	6.266	6.281	6.295	1 3 4	6 7 9	10 11 13
.80	6.310	6.324	6.339	6.353	6.368	6.383	6.397	6.412	6.427	6.442	1 3 4	6 7 9	10 12 13
.81	6.457	6.471	6.486	6.501	6.516	6.531	6.546	6.561	6.577	6.592	2 3 5	6 8 9	11 12 14
.82	6.607	6.622	6.637	6.653	6.668	6.683	6.699	6.714	6.730	6.745	2 3 5	6 8 9	11 12 14
.83	6.761	6.776	6.792	6.808	6.823	6.839	6.855	6.871	6.887	6.902	2 3 5	6 8 9	11 13 14
.84	6.918	6.934	6.950	6.966	6.982	6.998	7.015	7.031	7.047	7.063	2 3 5	6 8 10	11 13 15
.85	7.079	7.096	7.112	7.129	7.145	7.161	7.178	7.194	7.211	7.228	2 3 5	7 8 10	12 13 15
.86	7.244	7.261	7.278	7.295	7.311	7.328	7.345	7.362	7.379	7.396	2 3 5	7 8 10	12 13 15
.87	7.413	7.430	7.447	7.464	7.482	7.499	7.516	7.534	7.551	7.568	2 3 5	7 9 10	12 14 16
.88	7.586	7.603	7.621	7.638	7.656	7.674	7.691	7.709	7.727	7.745	2 4 5	7 9 11	12 14 16
.89	7.762	7.780	7.798	7.816	7.834	7.852	7.870	7.889	7.907	7.925	2 4 5	7 9 11	13 14 16
.90	7.943	7.962	7.980	7.998	8.017	8.035	8.054	8.072	8.091	8.110	2 4 6	7 9 11	13 15 17
.91	8.128	8.147	8.166	8.185	8.204	8.222	8.241	8.260	8.279	8.299	2 4 6	8 9 11	13 15 17
.92	8.318	8.337	8.356	8.375	8.395	8.414	8.433	8.453	8.472	8.492	2 4 6	8 10 12	14 15 17
.93	8.511	8.531	8.551	8.570	8.590	8.610	8.630	8.650	8.670	8.690	2 4 6	8 10 12	14 16 18
.94	8.710	8.730	8.750	8.770	8.790	8.810	8.831	8.851	8.872	8.892	2 4 6	8 10 12	14 16 18
.95	8.913	8.933	8.954	8.974	8.995	9.016	9.036	9.057	9.078	9.099	2 4 6	8 10 12	15 17 19
.96	9.120	9.141	9.162	9.183	9.204	9.226	9.247	9.268	9.290	9.311	2 4 6	8 11 13	15 17 19
.97	9.333	9.354	9.376	9.397	9.419	9.441	9.462	9.484	9.506	9.528	2 4 7	9 11 13	15 17 20
.98	9.550	9.572	9.594	9.616	9.638	9.661	9.683	9.705	9.727	9.750	2 4 7	9 11 13	16 18 20
.99	9.772	9.795	9.817	9.840	9.863	9.886	9.908	9.931	9.954	9.977	2 5 7	9 11 14	16 18 20

Amount for 1
$(1 + i)^n$

n	$\frac{1}{2}\%$	1%	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	$3\frac{1}{2}\%$	n
1	1.00500	1.01000	1.01500	1.02000	1.02500	1.03000	1.03500	1
2	1.01003	1.02010	1.03023	1.04040	1.05063	1.06090	1.07123	2
3	1.01508	1.03030	1.04568	1.06121	1.07689	1.09273	1.10872	3
4	1.02015	1.04060	1.06136	1.08243	1.10381	1.12551	1.14752	4
5	1.02525	1.05101	1.07728	1.10408	1.13141	1.15927	1.18769	5
6	1.03038	1.06152	1.09344	1.12616	1.15969	1.19405	1.22926	6
7	1.03553	1.07214	1.10985	1.14869	1.18869	1.22987	1.27228	7
8	1.04071	1.08286	1.12649	1.17166	1.21840	1.26677	1.31681	8
9	1.04591	1.09369	1.14339	1.19509	1.24886	1.30477	1.36290	9
10	1.05114	1.10462	1.16054	1.21899	1.28008	1.34392	1.41060	10
11	1.05640	1.11567	1.17795	1.24337	1.31209	1.38423	1.45997	11
12	1.06168	1.12683	1.19562	1.26824	1.34489	1.42576	1.51107	12
13	1.06699	1.13809	1.21355	1.29361	1.37851	1.46853	1.56396	13
14	1.07232	1.14947	1.23176	1.31948	1.41297	1.51259	1.61869	14
15	1.07768	1.16097	1.25023	1.34587	1.44830	1.55797	1.67535	15
16	1.08307	1.17258	1.26899	1.37279	1.48451	1.60471	1.73399	16
17	1.08849	1.18430	1.28802	1.40024	1.52162	1.65285	1.79468	17
18	1.09393	1.19615	1.30734	1.42825	1.55966	1.70243	1.85749	18
19	1.09940	1.20811	1.32695	1.45681	1.59865	1.75351	1.92250	19
20	1.10490	1.22019	1.34686	1.48595	1.63862	1.80611	1.98979	20
21	1.11042	1.23239	1.36706	1.51567	1.67958	1.86029	2.05943	21
22	1.11597	1.24472	1.38756	1.54598	1.72157	1.91610	2.13151	22
23	1.12155	1.25716	1.40838	1.57690	1.76461	1.97359	2.20611	23
24	1.12716	1.26973	1.42950	1.60844	1.80873	2.03279	2.28333	24
25	1.13280	1.28243	1.45095	1.64061	1.85394	2.09378	2.36324	25
26	1.13846	1.29526	1.47271	1.67342	1.90029	2.15659	2.44596	26
27	1.14415	1.30821	1.49480	1.70689	1.94780	2.22129	2.53157	27
28	1.14987	1.32129	1.51722	1.74102	1.99650	2.28793	2.62017	28
29	1.15562	1.33450	1.53998	1.77584	2.04641	2.35657	2.71188	29
30	1.16140	1.34785	1.56308	1.81136	2.09757	2.42726	2.80679	30
31	1.16721	1.36133	1.58653	1.84759	2.15001	2.50008	2.90503	31
32	1.17304	1.37494	1.61032	1.88454	2.20376	2.57508	3.00671	32
33	1.17891	1.38869	1.63448	1.92223	2.25885	2.65234	3.11194	33
34	1.18480	1.40258	1.65900	1.96068	2.31532	2.73191	3.22086	34
35	1.19073	1.41660	1.68388	1.99989	2.37321	2.81386	3.33359	35
36	1.19668	1.43077	1.70914	2.03989	2.43254	2.89828	3.45027	36
37	1.20266	1.44508	1.73478	2.08069	2.49335	2.98523	3.57103	37
38	1.20868	1.45953	1.76080	2.12230	2.55568	3.07478	3.69601	38
39	1.21472	1.47412	1.78721	2.16474	2.61957	3.16703	3.82537	39
40	1.22079	1.48886	1.81402	2.20804	2.68506	3.26204	3.95926	40
41	1.22690	1.50375	1.84123	2.25220	2.75219	3.35990	4.09783	41
42	1.23303	1.51879	1.86885	2.29724	2.82100	3.46070	4.24126	42
43	1.23920	1.53398	1.89688	2.34319	2.89152	3.56452	4.38970	43
44	1.24539	1.54932	1.92533	2.39005	2.96381	3.67145	4.54334	44
45	1.25162	1.56481	1.95424	2.43785	3.03790	3.78160	4.70236	45
46	1.25788	1.58046	1.98353	2.48661	3.11385	3.89504	4.86694	46
47	1.26417	1.59626	2.01328	2.53634	3.19170	4.01190	5.03728	47
48	1.27049	1.61223	2.04348	2.58707	3.27149	4.13225	5.21359	48
49	1.27684	1.62835	2.07413	2.63881	3.35328	4.25622	5.39606	49
50	1.28323	1.64463	2.10524	2.69159	3.43711	4.38391	5.58493	50
n	$\frac{1}{2}\%$	1%	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	$3\frac{1}{2}\%$	n

Amount for 1

$$(1+i)^n$$

n	4%	5%	6%	7%	8%	9%	10%	n
1	1.04000	1.05000	1.06000	1.07000	1.08000	1.09000	1.10000	1
2	1.08160	1.10250	1.12360	1.14490	1.16640	1.18810	1.21000	2
3	1.12486	1.15763	1.19102	1.22504	1.25971	1.29503	1.33100	3
4	1.16986	1.21551	1.26248	1.31080	1.36049	1.41158	1.46410	4
5	1.21665	1.27628	1.33823	1.40255	1.46933	1.53862	1.61051	5
6	1.26532	1.34010	1.41852	1.50073	1.58687	1.67710	1.77156	6
7	1.31593	1.40710	1.50363	1.60578	1.71382	1.82804	1.94872	7
8	1.36857	1.47746	1.59385	1.71819	1.85093	1.99256	2.14359	8
9	1.42331	1.55133	1.68948	1.83846	1.99900	2.17189	2.35795	9
10	1.48024	1.62889	1.79085	1.96715	2.15892	2.36736	2.59374	10
11	1.53945	1.71034	1.89830	2.10485	2.33164	2.58043	2.85312	11
12	1.60103	1.79586	2.01220	2.25219	2.51817	2.81266	3.13843	12
13	1.66507	1.88565	2.13293	2.40985	2.71962	3.06580	3.45227	13
14	1.73168	1.97993	2.26090	2.57853	2.93719	3.34173	3.79750	14
15	1.80094	2.07893	2.39656	2.75903	3.17217	3.64248	4.17725	15
16	1.87298	2.18287	2.54035	2.95216	3.42594	3.97031	4.59497	16
17	1.94790	2.29202	2.69277	3.15882	3.70002	4.32763	5.05447	17
18	2.02582	2.40662	2.85434	3.37993	3.99602	4.71712	5.55992	18
19	2.10685	2.52695	3.02560	3.61653	4.31570	5.14166	6.11591	19
20	2.19112	2.65330	3.20714	3.86968	4.66096	5.60441	6.72750	20
21	2.27877	2.78596	3.39956	4.14056	5.03383	6.10881	7.40025	21
22	2.36992	2.92526	3.60354	4.43040	5.43654	6.65860	8.14027	22
23	2.46472	3.07152	3.81975	4.74053	5.87146	7.25787	8.95430	23
24	2.56330	3.22510	4.04893	5.07237	6.34118	7.91108	9.84973	24
25	2.66584	3.38635	4.29187	5.42743	6.84848	8.62308	10.83471	25
26	2.77247	3.55567	4.54938	5.80735	7.39635	9.39916	11.91818	26
27	2.88337	3.73346	4.82235	6.21387	7.98806	10.24508	13.10999	27
28	2.99870	3.92013	5.11169	6.64884	8.62711	11.16714	14.42099	28
29	3.11865	4.11614	5.41839	7.11426	9.31727	12.17218	15.86309	29
30	3.24340	4.32194	5.74349	7.61226	10.06266	13.26768	17.44940	30
31	3.37313	4.53804	6.08810	8.14511	10.86767	14.46177	19.19434	31
32	3.50806	4.76494	6.45339	8.71527	11.73708	15.76333	21.11378	32
33	3.64838	5.00319	6.84059	9.32534	12.67605	17.18203	23.22515	33
34	3.79432	5.25335	7.25103	9.97811	13.69013	18.72841	25.54767	34
35	3.94609	5.51602	7.68609	10.67658	14.78534	20.41397	28.10244	35
36	4.10393	5.79182	8.14725	11.42394	15.96817	22.25123	30.91268	36
37	4.26809	6.08141	8.63609	12.22362	17.24563	24.25384	34.00395	37
38	4.43881	6.38548	9.15425	13.07927	18.62528	26.43668	37.40434	38
39	4.61637	6.70475	9.70351	13.99482	20.11530	28.81598	41.14478	39
40	4.80102	7.03999	10.28572	14.97446	21.72452	31.40942	45.25926	40
41	4.99306	7.39199	10.90286	16.02267	23.46248	34.23627	49.78518	41
42	5.19278	7.76159	11.55703	17.14426	25.33948	37.31753	54.76370	42
43	5.40050	8.14967	12.25045	18.34435	27.36664	40.67611	60.24007	43
44	5.61652	8.55715	12.98548	19.62846	29.55597	44.33696	66.26408	44
45	5.84118	8.98501	13.76461	21.00245	31.92045	48.32729	72.89048	45
46	6.07482	9.43426	14.59049	22.47262	34.47409	52.67674	80.17953	46
47	6.31782	9.90597	15.46592	24.04571	37.23201	57.41765	88.19749	47
48	6.57053	10.40127	16.39387	25.72891	40.21057	62.58524	97.01723	48
49	6.83335	10.92133	17.37750	27.52993	43.42742	68.21791	106.71896	49
50	7.10668	11.46740	18.42015	29.45703	46.90161	74.35752	117.39085	50
n	4%	5%	6%	7%	8%	9%	10%	n

SQUARES AND SQUARE ROOTS

n	n^2	$\sqrt{n}$	n	n^2	$\sqrt{n}$	n	n^2	$\sqrt{n}$	n	n^2	$\sqrt{n}$
1	1	1.000	**51**	2601	7.141	**101**	10 201	10.050	**151**	22 801	12.288
2	4	1.414	**52**	2704	7.211	**102**	10 404	10.100	**152**	23 104	12.329
3	9	1.732	**53**	2809	7.280	**103**	10 609	10.149	**153**	23 409	12.369
4	16	2.000	**54**	2916	7.348	**104**	10 816	10.198	**154**	23 716	12.410
5	25	2.236	**55**	3025	7.416	**105**	11 025	10.247	**155**	24 025	12.450
6	36	2.449	**56**	3136	7.483	**106**	11 236	10.296	**156**	24 336	12.490
7	49	2.646	**57**	3249	7.550	**107**	11 449	10.344	**157**	24 649	12.530
8	64	2.828	**58**	3364	7.616	**108**	11 664	10.392	**158**	24 964	12.570
9	81	3.000	**59**	3481	7.681	**109**	11 881	10.440	**159**	25 281	12.610
10	100	3.162	**60**	3600	7.746	**110**	12 100	10.488	**160**	25 600	12.649
11	121	3.317	**61**	3721	7.810	**111**	12 321	10.536	**161**	25 921	12.689
12	144	3.464	**62**	3844	7.874	**112**	12 544	10.583	**162**	26 244	12.728
13	169	3.606	**63**	3969	7.937	**113**	12 769	10.630	**163**	26 569	12.767
14	196	3.742	**64**	4096	8.000	**114**	12 996	10.677	**164**	26 896	12.806
15	225	3.873	**65**	4225	8.062	**115**	13 225	10.724	**165**	27 225	12.845
16	256	4.000	**66**	4356	8.124	**116**	13 456	10.770	**166**	27 556	12.884
17	289	4.123	**67**	4489	8.185	**117**	13 689	10.817	**167**	27 889	12.923
18	324	4.243	**68**	4624	8.246	**118**	13 924	10.863	**168**	28 224	12.961
19	361	4.359	**69**	4761	8.307	**119**	14 161	10.909	**169**	28 561	13.000
20	400	4.472	**70**	4900	8.367	**120**	14 400	10.954	**170**	28 900	13.038
21	441	4.583	**71**	5041	8.426	**121**	14 641	11.000	**171**	29 241	13.077
22	484	4.690	**72**	5184	8.485	**122**	14 884	11.045	**172**	29 584	13.115
23	529	4.796	**73**	5329	8.544	**123**	15 129	11.091	**173**	29 929	13.153
24	576	4.899	**74**	5476	8.602	**124**	15 376	11.136	**174**	30 276	13.191
25	625	5.000	**75**	5625	8.660	**125**	15 625	11.180	**175**	30 625	13.229
26	676	5.099	**76**	5776	8.718	**126**	15 876	11.225	**176**	30 976	13.267
27	729	5.196	**77**	5929	8.775	**127**	16 129	11.269	**177**	31 329	13.304
28	784	5.292	**78**	6084	8.832	**128**	16 384	11.314	**178**	31 684	13.342
29	841	5.385	**79**	6241	8.888	**129**	16 641	11.358	**179**	32 041	13.379
30	900	5.477	**80**	6400	8.944	**130**	16 900	11.402	**180**	32 400	13.416
31	961	5.568	**81**	6561	9.000	**131**	17 161	11.446	**181**	32 761	13.454
32	1024	5.657	**82**	6724	9.055	**132**	17 424	11.489	**182**	33 124	13.491
33	1089	5.745	**83**	6889	9.110	**133**	17 689	11.533	**183**	33 489	13.528
34	1156	5.831	**84**	7056	9.165	**134**	17 956	11.576	**184**	33 856	13.565
35	1225	5.916	**85**	7225	9.220	**135**	18 225	11.619	**185**	34 225	13.601
36	1296	6.000	**86**	7396	9.274	**136**	18 496	11.662	**186**	34 596	13.638
37	1369	6.083	**87**	7569	9.327	**137**	18 769	11.705	**187**	34 969	13.675
38	1444	6.164	**88**	7744	9.381	**138**	19 044	11.747	**188**	35 344	13.711
39	1521	6.245	**89**	7921	9.434	**139**	19 321	11.790	**189**	35 721	13.748
40	1600	6.325	**90**	8100	9.487	**140**	19 600	11.832	**190**	36 100	13.784
41	1681	6.403	**91**	8281	9.539	**141**	19 881	11.874	**191**	36 481	13.820
42	1764	6.481	**92**	8464	9.592	**142**	20 164	11.916	**192**	36 864	13.856
43	1849	6.557	**93**	8649	9.644	**143**	20 449	11.958	**193**	37 249	13.892
44	1936	6.633	**94**	8836	9.695	**144**	20 736	12.000	**194**	37 636	13.928
45	2025	6.708	**95**	9025	9.747	**145**	21 025	12.042	**195**	38 025	13.964
46	2116	6.782	**96**	9216	9.798	**146**	21 316	12.083	**196**	38 416	14.000
47	2209	6.856	**97**	9409	9.849	**147**	21 609	12.124	**197**	38 809	14.036
48	2304	6.928	**98**	9604	9.899	**148**	21 904	12.166	**198**	39 204	14.071
49	2401	7.000	**99**	9801	9.950	**149**	22 201	12.207	**199**	39 601	14.107
50	2500	7.071	**100**	10 000	10.000	**150**	22 500	12.247	**200**	40 000	14.142

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Cover photo, courtesy of IBM Canada Limited, of a two-layer memory module as used in the IBM family of computers. The technology, developed by IBM, involves the mounting of four tiny memory chips on ceramic modules, about 1 cm square. Each memory chip can contain 174 complete circuits. Some circuits switch on and off in billionths of a second.

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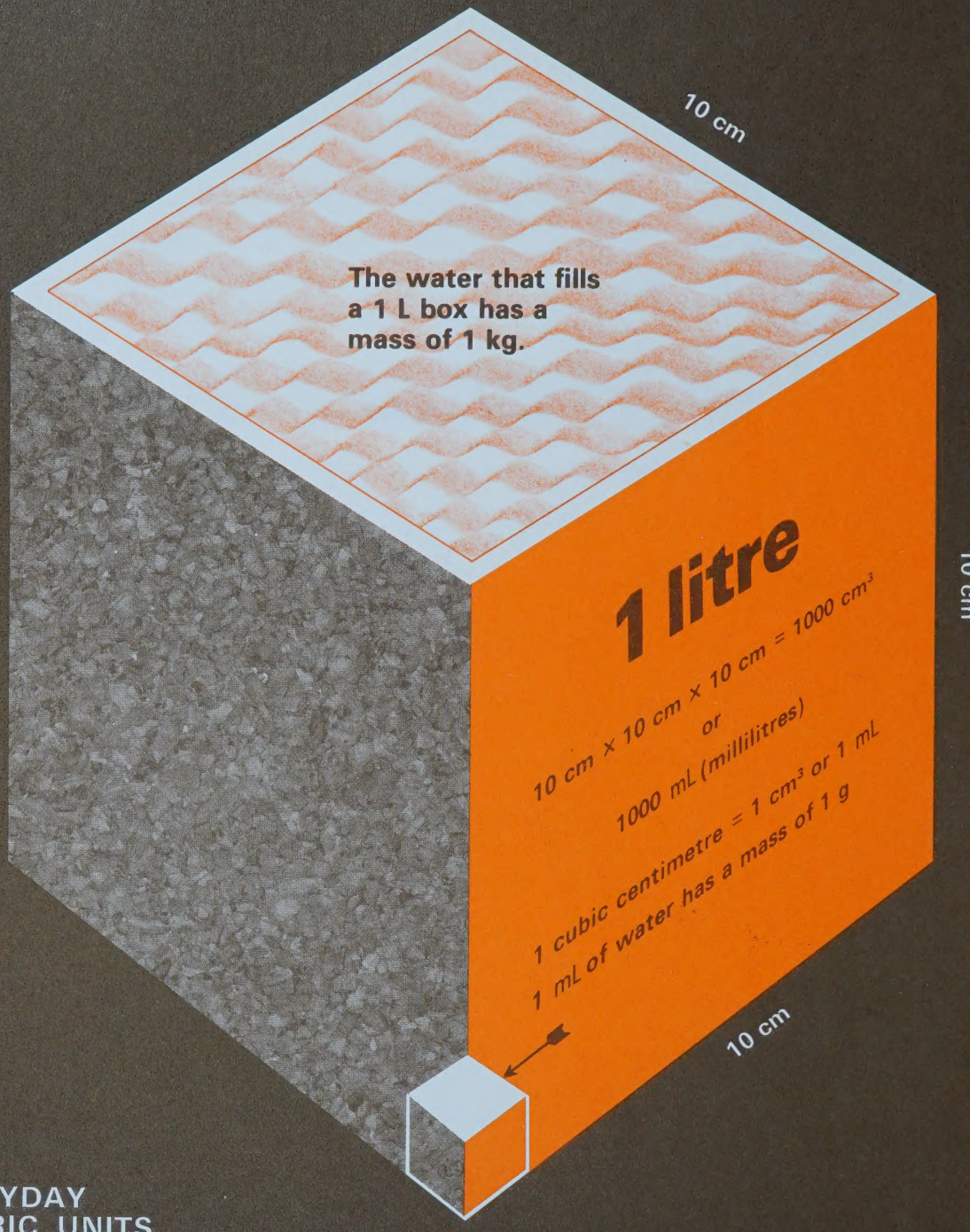
10 cm
The water that fills a 1 L box has a mass of 1 kg.
1 litre
10 cm x 10 cm x 10 cm = 1000 cm³
or
1000 mL (millilitres)
1 cubic centimetre = 1 cm³ or 1 mL
1 mL of water has a mass of 1 g
10 cm
10 cm

EVERYDAY METRIC UNITS

Length		*Area*		*Volume*		*Mass*	
millimetre	mm	square centimetre	cm^2	cubic centimetre	cm^3 (or mL)	gram	g
centimetre	cm	square metre	m^2	litre (1000 cm^3)	litre or L	kilogram	kg
metre	m	hectare (10 000 m^2)	ha	kilolitre	kL	tonne	t
kilometre	km	square kilometre	km^2				

SI Base Units

Quantity	*Unit*	*Symbol*
length	metre	m
mass	kilogram	kg
time	second	s
temperature	kelvin*	K
current	ampere	A
luminous intensity	candela	cd
amount of substance	mole	mol

*degrees Celsius (°C) for everyday use

SI Prefixes

Prefix	*Multiplication Factor*		*Symbol*
tera	1 000 000 000 000	$= 10^{12}$	T
giga	1 000 000 000	$= 10^{9}$	G
mega	1 000 000	$= 10^{6}$	M
kilo	1 000	$= 10^{3}$	k
hecto	100	$= 10^{2}$	h
deca	10	$= 10^{1}$	da
deci	0.1	$= 10^{-1}$	d
centi	0.01	$= 10^{-2}$	c
milli	0.001	$= 10^{-3}$	m
micro	0.000 001	$= 10^{-6}$	μ
nano	0.000 000 001	$= 10^{-9}$	n
pico	0.000 000 000 001	$= 10^{-12}$	p

SI Derived Units

Quantity	*Unit*	*Symbol*	*Formula*
area	square metre	m^2	
volume	cubic metre	m^3	
density	kilogram per cubic metre	kg/m^3	
frequency	hertz	Hz	1/s
velocity, linear	metre per second	m/s	
acceleration, linear	metre per second per second	m/s^2	
force	newton	N	$kg \cdot m/s^2$
energy	joule	J	N·m
power	watt	W	J/s
pressure	pascal	Pa	N/m^2
electric charge	coulomb	C	A·s
potential difference	volt	V	W/A
electric resistance	ohm	Ω	V/A
luminous flux	lumen	lm	cd·sr
illumination	lux	lx	lm/m^2

SI Conventions

- No period after the symbol, except to end a sentence.
- Same symbol used for singular and plural.
- Most symbols are small letters. Capitals are used when the unit is named after a famous person, as N for the newton.
- Units of area are written cm^2, m^2, and volume cm^3, m^3.
- Use a space instead of a comma for numbers of more than four digits, as 10 000 m or 0.000 75 mm.

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